Autumn Semester 2014–2015

MATH31051/MATH41051/MATH61051 *Introduction to Topology*

January 2015 Examination Solutions and Feedback

A1. (a) A *topology* τ on a set X is a collection of subsets of X (the *open sets* of the topology) such that

(i) $\emptyset$, $X \in \tau$;

(ii) $U_1, U_2 \in \tau \Rightarrow U_1 \cap U_2 \in \tau$;

(iii) $U_\lambda \in \tau$ for $\lambda \in \Lambda \Rightarrow \bigcup_{\lambda \in \Lambda} U_\lambda \in \tau$.

[3 marks]

(b) $f\colon X \to Y$ is *continuous* when

$$V \text{ open in } Y \Rightarrow f^{-1}(V) \text{ open in } X.$$

f is a *homeomorphism* when it is a continuous bijection with a continuous inverse.

[3 marks]

(c) Define $f\colon X \to S^1 \times [0,1]$ by

$$f(\boldsymbol{x}) = (\boldsymbol{x}/|\boldsymbol{x}|, |\boldsymbol{x}| - 1).$$

The inverse $g\colon S^1 \times [0,1] \to X$ is given by

$$g(\boldsymbol{y}, t) = (1+t)\boldsymbol{y}.$$

Each of these functions is continuous since the component functions are standard continuous functions. Hence $X \cong S^1 \times [0,1]$. **[4 marks]**

It was surprising that several people were unable to reproduce this very basic definitions. Common errors were the following.

- *Saying that a topology was 'a collection of open subsets of X. The open subsets are the subsets in the topology but the notion of 'open' does not in general mean anything until you have a topology.*
- *Only including finite unions in condition (iii) for a topology.*
- *Writing $\Leftrightarrow$ instead of $\Rightarrow$ in the definition of continuity.*
- *Giving the epsilon delta definition of continuity (only a few people did this).*

I know that many students find the contruction of functions as is required in part (c) of this question difficult. You do need to make sure that the defined functions finish up in the right space.

A2. (a) A *path* in X from x_0 to x_1 is a continuous function $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$. [**2 marks**]

(b) We may define an equivalence relation $\sim$ on a topological space X by $x \sim x'$ if there is a path in X from x to x'. Then the *path-components* of X are the equivalence classes of this equivalence relation. [**2 marks**]

(c) $p \in X$ is a *cut-point of type* n when $X \setminus \{p\}$ has n path-components. A pair of distinct points $\{p, q\}$ in X is a *cut-pair of type* n when $X \setminus \{p, q\}$ has n path-components.

[**2 marks**]

(d) It can be proved that a homeomorphism between topological spaces induces a bijection betwen the cut-points and cut-pairs of each type.

Set (ii) has a cut-points of type 2 (for example the mid-point of the line segment) whereas (i) and (iii) have only cut-points of type 1. This shows that (ii) is not homeomorphic to (i) or (iii).

Set (i) has only cut-pairs of type 2 whereas (iii) has cut-pairs of type 1 (for example one point on the upper semicircle and one on the lower semicircle) and so (i) is not homeomorphic to (iii). [**4 marks**]

Some students missed out the word 'continuous' of the definition in part (a). Some students failed to say what the equivalence relation is in part (b). Quite a lot of students failed to use the word 'path-components' in the definition in part (c) even though they had just defined it. In part (d) some students claimed that a bijection between cut-points of each type was a sufficient condition for a continuous function to be a homeomorphism: it is a necessary condition but not sufficient.

A3. (a) X is *Hausdorff* when for each pair of distinct points x_1, $x_2 \in X$, there are open sets U_1 and U_2 in X such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 \neq \emptyset$. [**2 marks**]

(b) This space is not Hausdorff because every open set containing a also contains c and so open sets as required cannot be found for $x_1 = a$ and $x_2 = c$.

[**2 marks**]

(c) The *product topology* on $X \times Y$ has a basis $\{\, U \times V \mid U$ open in X, V open in $Y \,\}$, i.e. the open sets consist of all unions of such sets. [**2 marks**]

(d) Suppose X and Y are Hausdorff spaces and $(x_1, y_1) \neq (x_2, y_2) \in X \times Y$. Then $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$ then there are disjoint open subsets U_1, $U_2 \subset X$ such that $x_1 \in U_1$ and $x_2 \in U_2$. Then $U_1 \times Y$, $U_2 \times Y$ are disjoint open subsets of $X \times Y$ as required. There is a similar argument if $y_1 \neq y_2$. Hence $X \times Y$ is Hausdorff. [**4 marks**]

A common error in (a) was to fail to say that the points are distinct. Some students got the quantifiers wrong in the definition in so only required the existence of a pair of points with the property. A common error in (c) was the say that the open sets in $X \times Y$ all have the form $U \times V$: a general open set is a union of such sets. In part (d) quite a lot of students only considered the case when $x_1 \neq x_2$ and $y_1 \neq y_2$ whereas in general only one of these is true when $(x_1, y_1) \neq (x_2, y_2)$.

A4. (a) Two paths σ_0 and $\sigma_1 \colon I \to X$ from x_0 to x_1 are *homotopic* if there is a continuous map (a homotopy) $H \colon I^2 \to X$ such that

$$\begin{aligned} H(s,0) &= \sigma_0(s), \\ H(s,1) &= \sigma_1(s), \\ H(0,t) &= x_0, \\ H(1,t) &= x_1 \end{aligned}$$

for all $s,\, t \in I$. **[3 marks, bookwork]**

(b) Given two paths σ and τ in X, the *product* $\sigma * \tau$ is defined when $\sigma(1) = \tau(0)$ and is given by

$$\sigma * \tau(s) = \begin{cases} \sigma(2s) & \text{for } 0 \leqslant s \leqslant 1/2, \\ \tau(2s-1) & \text{for } 1/2 \leqslant s \leqslant 1 \end{cases}$$

for $s \in I$. This function is well-defined by the condition on σ and τ and is continuous by the Gluing Lemma. **[3 marks, bookwork]**

(c) From the data, the terminal point of σ is the same as the initial point of τ and the terminal point of τ is the same as the initial point of ρ and so the products in the question are defined. The required homotopy is given by

$$H(s,t) = \begin{cases} \sigma\big(4s/(1+t)\big) & \text{for } 0 \leqslant s \leqslant (1+t)/4, \\ \tau\big(4(s-(1+t)/4)\big) & \text{for } (1+t)/4 \leqslant s \leqslant (2+t)/4, \\ \rho\big((s-(2+t)/4)/(1-(2+t)/4\big) & \text{for } (2+t)/4 \leqslant s \leqslant 1. \end{cases}$$

This function is well-defined and is continuous by the Gluing Lemma.

[4 marks, bookwork]

A common error in (a) was to fail to say that H is continuous. Quite a lot of students struggled with part (c). A diagram of the homotopy earned some credit.

B5. (a) Let $q \colon X \to X/{\sim}$ be given by $q(x) = [x]$, the equivalence class of x. Then the *quotient topology* on $X/{\sim}$ is given by $V \subset X/{\sim}$ is open in $X/{\sim}$ if and only if $q^{-1}(V)$ is open in X.

The *universal property* of the quotient topology is: given a topological space Y, a function $F \colon X/{\sim} \to Y$ is continuous if and only if $F \circ q \colon X \to Y$ is continuous.

[4 marks]

(b) Given a continuous surjection $f \colon X \to Y$, define an equivalence relation on X by $x \sim x' \Leftrightarrow f(x) = f(x')$. Then we may define $F \colon X/{\sim} \to Y$ by $F([x]) = f(x)$.
Since $[x] = [x'] \Rightarrow x \sim x' \Rightarrow f(x) = f(x')$ (by the definition of the equivalence relation), the function F is well-defined.
Since $F([x]) = F([x']) \Rightarrow f(x) = f(x') \Rightarrow x \sim x'$ (by the definition of the equivalence relation) $\Rightarrow [x] = [x']$, F is a monomorphism.
Since f is a surjection, $y = f(x)$ for some $x \in X$ and so $y = F([x])$. Hence F is a surjection.
This shows that $F \colon X/{\sim}\;\; Y$ is a bijection.

The function $F \colon X/{\sim} \to Y$ is continuous by the universal property since $F \circ q = f$ which is given as continuous.

The space $X/{\sim} = q(X)$ is compact since it is the continuous image of a compact set. Hence F is a homeomorphism since it is a continuous bijection from a compact space to a Hausdorff space. **[7 marks]**

(c) Define $f \colon S^1 \times [0,1] \to D^2$ by $f(\boldsymbol{x}, t) = (1-t)\boldsymbol{x}$. This is a continuous surjection. Furthermore $f(x,t) = f(x',t')$ if and only if $(x,t) = (x',t')$ or $t = t' = 1$, i.e. $(x,t) = (x',t')$, or (x,t) and

$(x', t') \in S^1 \times \{1\}$. Furthermore, $S^1 \times [0,1]$ is compact (a product of closed bounded subsets of Euclidean spaces) and D^2 is Hausdorff (a subspace of a Euclidean space). Thus the general result in (b) gives the required homeomophism $F\colon (S^1 \times [0,1])/(S^1 \times \{1\} \to D^2$.

[4 marks]

In part (a) students often described the universal property without referring to the function $q\colon X \to X/\sim$ earlier in the question. The question is about the quotient topology which has just been defined. There were quite a lot of poor answers to part (b) in spite of the large number of examples of this argument done in the course. Many students failed to define the equivalence relation and just used it without saying what it was. This question was simply asking for a proof of a result in the course. I was less surprised that many students found part (c) tricky since I know that as in Question 1 many find the construction of maps difficult.

B6. (a) A subset K of a topological space X is *compact* if each cover for K by open sets of X has a finite subcover for K.

If X itself is a compact subset then X is a *compact space* **[3 marks]**

(b) Suppose that $\mathcal{F}$ is an open cover for $f(K)$. Then the set $f^{-1}(\mathcal{F}) = \{ f^{-1}(U) \mid U \in \mathcal{F} \}$ is an open cover for K since given $x \in K$, $f(x) \in f(K)$ and so $f(x) \in U$ for some $U \in \mathcal{F}$, i.e. $x \in f^{-1}(U)$. Hence $f^{-1}(\mathcal{F})$ has a finite subcover $\{ f^{-1}(U_1), f^{-1}(U_2), \ldots, f^{-1}(U_n) \}$ for K. Hence $\{ U_1, U_2, \ldots, U_n \}$ is a finite subcover of $\mathcal{F}$ for $f(K)$ as required to prove that $f(K)$ is compact. **[5 marks]**

(c) Suppose for contradiction that the subsets A_n are as in the question but that $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Then $\bigcup_{n=1}^{\infty}(X \setminus A_n) = X \setminus \bigcap_{n=1}^{\infty} A_n = X$ and so $\{ X \setminus A_n \mid n \geqslant 1 \}$ is an open cover for X. But now, since X is a compact space, there must be a finite subcover for X. Now notice that the nesting of the subsets A_n means that $X \setminus A_1 \subset X \setminus A_2 \subset \cdots \subset X \setminus A_n \subset X \setminus A_{n+1} \subset \cdots$. Let $X \setminus A_k$ be the largest subset in the finite subcover. Then $X \subset X \setminus A_k$ and so $A_k = \emptyset$ contradicting the choice of the sets A_n as non-empty subsets. Hence, $\bigcap_{n=1}^{\infty} A_n$ is non-empty as required. **[7 marks]**

In (a) many students stated that a compact topological space is one for which every subset is compact. I don't know where this came from for it is not an error I have ever seen before. In (b), you have to start with an open cover of $f(K)$. Quite a lot of students went from a finite open covering for K, say $\{V_1, \ldots, V_n\}$, to suggest that $\{ f(V_1), \ldots, f(V_n)\}$ is an open covering for $f(K)$ which is not correct since the sets $f(V_i)$ may not be open. Quite a lot of students did not seem to consider that part (c) might follow on from the previous parts and so didn't mention coverings at all.

B7. (a) The *fundamental group* of X based at x_0, $\pi_1(X, x_0)$ is the set of equivalence classes of loops in X based at x_0 (i.e. paths in X from x_0 to x_0) with multiplication given by $[\sigma][\tau] = [\sigma * \tau]$.

The group product is well-defined since, if $\sigma_0 \sim \sigma_1$ and $\tau_0 \sim \tau_1$ are pairs of homotopic loops based at x_0 then $\sigma_0 * \tau_0 \sim \sigma_1 * \tau_1$.

The group product is associative since $(\sigma * \tau) * \rho \sim \sigma * (\tau * \rho)$ for loops σ, τ and ρ based at x_0.

The group identity is $[\varepsilon_{x_0}]$, the class of the constant loop, since $\varepsilon_{x_0} * \sigma \sim \sigma \sim \sigma * \varepsilon_{x_0}$.

The inverse of σ is given by the reverse loop $\overline{\sigma}$ since $\sigma * \overline{\sigma} \sim \varepsilon_{x_0} \sim \overline{\sigma} * \sigma$.

[6 marks]

(b) Given a path ρ from x_0 to x_1, we define $u_\rho([\sigma]) = [\overline{\rho}][\sigma][\rho]$ for all $[\sigma] \in \pi_1(X, x_0)$.

This is a homomorphism since

$$u_\rho([\sigma]u_\rho([\tau]) = [\overline{\rho}][\sigma][\rho][\overline{\rho}][\tau][\rho] = [\overline{\rho}][\sigma][\varepsilon_{x_0}][\tau][\rho] = [\overline{\rho}][\sigma][\tau][\rho] = u_\rho([\sigma][\tau]).$$

It is an isomorphism since it has an inverse given by $u_{\overline{\rho}}$ since

$$u_{\overline{\rho}}u_\rho([\sigma]) = [\rho][\overline{\rho}][\sigma][\rho][\overline{\rho}] = [\varepsilon_{x_0}][\sigma][\varepsilon_{x_0}] = [\sigma]$$

and similarly $u_\rho \circ u_{\overline{\rho}} = 1$. **[4 marks]**

(c) Suppose that all paths from x_0 to x_1 induce the same isomorphism and that σ and τ are two loops based at x_0. Let ρ be a path in X from x_0 to x_1. Then $\tau * \rho$ is a second path from x_0 to x_1 so that $u_{\tau*\rho}([\sigma]) = u_\rho([\sigma])$. This gives us $([\tau][\rho])^{-1}[\sigma]([\tau][\rho]) = [\rho]^{-1}[\sigma][\rho]$ which gives $[\rho]^{-1}[\tau]^{-1}[\sigma][\tau][\rho] = [\rho]^{-1}[\sigma][\rho]$ which gives $[\sigma][\tau] = [\tau][\sigma]$ proving that $\pi_1(X)$ is abelian. **[5 marks]**

In part (a) some students defined an element of the fundamental group to be a loop rather than a homotopy class of loops. Other students did not understand what 'well-defined' means — independence of the choice of represenatatives of the homotopy classes. In part (c) quite a lot of students proved that the homotopy classes $[\rho_1 * \overline{\rho}_2]$ and σ commute where σ is a loop based at x_0 and ρ_1 and ρ_2 are two paths from x_0 to x_1 but did not prove that every element of $\pi_1(X, x_0)$ could be written in the form $[\rho_1 * \overline{\rho}_2]$.

B8. (a) X_1 is a *retract* of X when there is a continuous map $r\colon X \to X_1$ such that $r(x) = x$ for $x \in X_1$. **[1 mark]**

(b) Define $r\colon \mathbb{R}^2 \setminus \{\mathbf{0}\}$ to S^1 by $r(\boldsymbol{x}) = \boldsymbol{x}/|\boldsymbol{x}|$. Then for $\boldsymbol{x} \in S^1$, $|\boldsymbol{x}| = 1$ and so $r(\boldsymbol{x}) = \boldsymbol{x}$. Hence r is a retraction and S^1 is a retract of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$. **[2 marks]**

(c) The function $f_*\colon \pi_1(X, x_0) \to \pi_1\big(Y, f(x_0)\big)$ is defined by $f_*[\sigma] = [f \circ \sigma]$ where σ is a loop in X based at x_0.

The function f_* is well-defined since, if $H\colon I^2 \to X$ is a homotopy between two loops σ_0 and σ_1 in X based at x_0 then $f \circ H$ is a homotopy between $f \circ \sigma_0$ and $f \circ \sigma_1$.

The function f_* is a homomorphism since $f_*([\sigma] \cdot [\tau]) = [f \circ (\sigma * \tau)] = [(f \circ \sigma) * (f \circ \tau)] = f_*([\sigma]) \cdot f_*([\tau])$. **[5 marks]**

(d) Suppose that X_1 is a retract of X with retraction map $r\colon X \to X_1$. Then $r \circ i = 1\colon X_1 \to X_1$ the identity map. Hence, by the functorial properties of the fundamental group, for $x_0 \in X_1$, $r_* \circ i_* = (r \circ i)_* = 1_* = 1\colon \pi_1(X_1, x_0) \to \pi_1(X, x_0) \to \pi_1(X_1, x_0)$. Hence, for α_1, $\alpha_2 \in \pi(X_1, x_0)$, $i_*(\alpha_1) = i_*(\alpha_2) \Rightarrow r_*i_*(\alpha_1) = r_*i_*(\alpha_2) \Rightarrow \alpha_1 = \alpha_2$ and so i_* is an injection and so a monomorphism. **[4 marks]**

(e) This may be used to prove that S^1 is not a retract of $\mathbb{R}^2$ since $\pi_1(S^1, x_0) \cong \mathbb{Z}$ whereas $\pi_1(\mathbb{R}^2, x_0) = \{1\}$, the trivial group and so the only function $\pi_1(S^1, x_0) \to \pi_1(D^2, x_0)$ is the constant function which is not a monomorphism. **[3 marks]**

The hint at the end should have suggested stating the homotopy groups of S^1 and $\mathbb{R}^2$ even if you got stuck in the question. This should have enabled anyone to do part (e) using the result asked for in (d) even if you had not been able to do (d).

The remaining questions were only on the MATH41051 and MATH61051 papers.

The problem students had with these three questions was that they had not really engaged with the additional reading.

C9. (a) $N \subset X$ is a *neighbourhood* of $x_0 \in N$ when there is an open set U such that $x_0 \in U \subset N$.

Suppose that $U \subset X$ is open. Then, for $x_0 \in U$, $x_0 \in U \subset U$ and so U is a neighbourhood of x_0.

Conversely, if U is a neighbourhood of each of its points, then, for each $x \in U$, there is an open set U_x such that $x \in U_x \subset U$. The $U = \bigcup_{x \in U} U_x$ is a union of open sets and so is open.

[**5 marks**]

(b) $f : X \to Y$ is continous at $x_0 \in X$ if, for each neighbourhood N of $f(x_0$ in Y, $f^{-1}(N)$ is a neighbourhood of x_0 in X.

Suppose that $f : X \to Y$ is a continuous function of topological spaces. Given $x \in X$ suppose that N is a neighbourhood of $f(x)$ in Y. Then, by definition, there is an open subset U of Y such that $f(x) \in U \subset N$. It follows that $x \in f^{-1}(U) \subset f^{-1}(N)$ and so, since $f^{-1}(U)$ is open in X (because f is continuous), $f^{-1}(N)$ is a neighbourhood of x as required to prove that f is continuous at x.

Conversely, suppose that $f : X \to Y$ is continuous at each $x \in X$ and suppose that V is an open subset of Y. Suppose that $x \in f^{-1}(V)$. Then $f(x) \in V$ and so V is a neighbourhood of $f(x)$ since V is open. Hence $f^{-1}(V)$ is a neighbourhood of x since f is continuous at x. Hence, $f^{-1}(V)$ is a neighbourhood of each of its points and so, by part (a), is open as required to prove that f is continuous. [**8 marks**]

(c) A point $x \in X$ is a *closure point* of $A \subset X$ if, for all neighbourhoods N of x, $N \cap A \neq \emptyset$. The set of closure points of A is called the *closure* of A. [**2 marks**]

(d) Suppose that $x \in \overline{A_1 \cap A_2}$. Then, for all neighbourhoods N of x, $N \cap (A_1 \cap A_2) \neq \emptyset$ and so $N \cap A_1 \neq \emptyset$ and $N \cap A_2 \neq \emptyset$. Hence $x \in \overline{A_1} \cap \overline{A_2}$. Hence $\overline{A_1 \cap A_2} \subset \overline{A_1} \cap \overline{A_2}$.

In the case that $A_1 = (0,1)$ and $A_2 = (1,2)$ in $\mathbb{R}$ with the usual topology, $\overline{A_1} = [0,1]$ and $\overline{A_2} = [1,2]$ and so $\overline{A_1} \cap \overline{A_2} = \{1\}$. However, $A_1 \cap A_2 = \emptyset$ and so $\overline{A_1 \cap A_2} = \emptyset$.

[**5 marks**]

C10. (a) Given a group G and a set X, then a *G-action* on the set X is a function $G \times X \to X$, written $(g,x) \mapsto g \cdot x$, such that $1 \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all g, $h \in G$ and $x \in X$ (where the group is written multiplicatively with identity 1). A topological space X is a G-space if there is a G-action $G \times X \to X$ such that the function $\theta_g : X \to X$ defined by $\theta_g(x) = g \cdot x$ is continuous for all $g \in G$.

For g, $h \in G$, $\theta_{gh}(x) = (gh) \cdot x = g \cdot (h \cdot x) = \theta_g\big(\theta_h(x)\big)$ and so $\theta_{gh} = \theta_g \circ \theta_h$. Also $\theta_1(x) = 1 \cdot x = x$ and so $\theta_1 = 1_X$, the identity function. Hence the function $\theta_g : X \to X$ is a homeomorphism since $\theta_{g^{-1}}$ is its inverse and so the function $g \mapsto \theta_g$ gives a homomorphism from G to the group of homeomorphisms $G \to G$. [**4 marks**]

(b) Given a G-space X, we may define an equivalence relation on X by $x \sim x' \Leftrightarrow x' = gx$ for some $g \in G$. The conditions on the G-action guarantee that this is an equivalence relation. We define $X/G = X/\!\sim$ with the quotient topology.

Suppose that $A \subset X$ is a closed subset. Then $q^{-1}q(A) = \{ x \in X \mid q(x) \in q(A) \} = \{ x \in X \mid x \sim a \text{ for some } a \in A \} = \cup_{g \in G} \theta_g(A)$ which is a finite union of closed sets (since G is finite and θ_g is a homeomorphism) and so is closed. Hence $q(A)$ is closed in $X/\!\sim$.

[**5 marks**]

(c) A topological space X is *normal* when for each pair of disjoint closed subsets A_i $(i = 1, 2)$ of X, there exists a disjoint pair of open subsets U_i $(i = 1, 2)$ of X such that $A_i \subset U_i$ for $i = 1, 2$.

Suppose that A and B are disjoint non-empty closed sets in a compact Hausdorff space X (the result is trivial if either A or B is empty). Let $a \in A$, then, for each $b \in B$ we can find disjoint open sets U_b and V_b such that $a \in U_b$ and $b \in V_b$. The set $\{ V_b \mid b \in B \}$ is an open cover for B. However B is a closed set in a compact space and so is compact (standard result from course). Hence there is a finite subcover $\{ V_{b_1}, \ldots, V_{b_n} \}$ for B. Put $W_a = U_{b_1} \cap \ldots \cap U_{b_n}$ and $W'_a = V_{b_1} \cup \ldots \cup V_{b_n}$. Then, by the choice of the points b_i, $B \subset W'_a$ and, since $a \in U_b$ for all $b \in B$, $a \in W_a$. Finally, since $U_b \cap V_b = \emptyset$, $W_a \cap V_b = \emptyset$ and so $W_a \cap W'_a = \emptyset$.

Now $\{W_a \mid a \in A\}$ is on open cover for A. Since A is closed in compact X then A is compact. Hence there is a finite subcover $\{W_{a_1}, \ldots, W_{a_m}\}$ for A. Put $U = W_{a_1} \cup \ldots \cup W_{a_m}$ and $V = W'_{a_1} \cap \ldots \cap W'_{a_m}$. Then, as above, $A \subset U$ and $B \subset V$ and $U \cap V = \emptyset$ as required.

[**9 marks**]

C11 (a) **Path-Lifting Theorem.** Suppose that $p\colon \widetilde{X} \to X$ is a covering and $\widetilde{x}_0 \in \widetilde{X}$, $x_0 \in X$ are points such that $p(\widetilde{x}_0) = x_0$. Then, given a loop $\gamma\colon I \to X$ based at x_0, there is a unique path $\widetilde{\gamma}\colon I \to \widetilde{X}$ such that $p\colon \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = \widetilde{x}_0$.

Monodromy Theorem. With the data in the previous theorem, if γ_0 and γ_1 are two equivalent loops in X based at x_0 with lifts $\widetilde{\gamma}_i$ such that $\widetilde{\gamma}_i(0) = \widetilde{x}_0$ $(i = 1, 2)$, then $\widetilde{\gamma}_0(1) = \widetilde{\gamma}_1(1)$.

[**4 marks**]

(b) Given a G-space where the action of G on X is properly discontinuous the quotient map $q\colon X \to X/G$ is a covering. So given a loop γ in X/G based at $[x_0]$, by the Path-Lifting Theorem, there is a path $\widetilde{\gamma}$ in X such that $q \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = x_0$. Then $q\widetilde{\gamma}(1) = \gamma(1) = [x_0]$ and so $\widetilde{\gamma}(1) = gx_0$ for some $g \in G$. We define $\phi\colon \pi_1(X/G, [x_0]) \to G$ by $\phi([\gamma]) = g$. This is well-defined by the Monodromy Theorem.

[**4 marks**]

(c) Let γ_1, γ_2 be two loops in X/G based at $[x_0]$ with lifts $\widetilde{\gamma}_i$ to X such that $\widetilde{\gamma}_i(0) = x_0$. Suppose that $\widetilde{\gamma}_i(1) = g_i x_0$ so that $\phi([\gamma_i]) = g_i$. Define $\rho\colon I \to X$ by

$$\rho(s) = \begin{cases} \widetilde{\gamma}_1(2s) & 0 \leqslant s \leqslant 1/2, \\ g_1\widetilde{\gamma}_2(2s-1) & 1/2 \leqslant s \leqslant 1. \end{cases}$$

This is well-defined and continuous by the Gluing Lemma and $q \circ \rho = \gamma_1 \star \gamma_2$. Hence, since $\rho(1) = g_1 g_2 x_0$, $\phi([\gamma_1][\gamma_2]) = \phi([\gamma_1 \star \gamma_2]) = g_1 g_2 = \phi([\gamma_1])\phi(\gamma_2])$ as required.

[**4 marks**]