Autumn Semester 2015–2016

MATH31051/MATH41051/MATH61051 *Introduction to Topology*

January 2016 Examination Solutions and Feedback

A1. (a) A *topology* τ on a set X is a collection of subsets of X (the *open subsets* of the topology) such that

(i) $\emptyset$, $X \in \tau$;

(ii) $U_1, U_2 \in \tau \Rightarrow U_1 \cap U_2 \in \tau$;

(iii) $U_\lambda \in \tau$ for $\lambda \in \Lambda$ (an arbitrary indexing set) $\Rightarrow \bigcup_{\lambda\in\Lambda} U_\lambda \in \tau$.

[*It is necessary to say what the element of the topology τ are (subsets of X). Also they cannot be required to be 'open subsets'. By definition an open subset of a topological space is a subset which is in the topology but 'open' has no meaning until you have defined the topology.*]

[**3 marks, bookwork**]

(b) $f\colon X \to Y$ is *continuous* when

$$V \text{ open in } Y \Rightarrow f^{-1}(V) \text{ open in } X.$$

[*Notice that this implication only goes one way.*]

f is a *homeomorphism* when it is a continuous bijection with a continuous inverse.

[**3 marks, bookwork**]

(c) Define $f\colon \{\, x \in \mathbb{R}^2 \mid 0 < |x| < 1 \,\} \to S^1 \times (1, 2)$ by $f(x) = (x/|x|, |x| + 1)$. The inverse of this map is given by $g(y, t) = (t-1)y$ and so f is a bijection. Both f and g are continuous since the component functions are standard continuous functions. Hence f is a homeomorphism and so $\{\, x \in \mathbb{R}^2 \mid 0 < |x| < 1 \,\}$ is homeomorphic to $S^1 \times (1, 2)$.

The first set is in $\mathbb{R}^2$ and the second in $\mathbb{R}^3$ so if the maps you write down don't land up in the right dimensional space they cannot possibly be right.]

[**4 marks, unseen but similar to exercises set**]

[**Total marks 10**]

A2. (a) A path in X from x_0 to x_1 is a continuous map $\sigma\colon I \to X$ (where $I = [0, 1]$ with the usual topology) such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. [**1 mark, bookwork**]

A topological space X is *path-connected* if, for each pair of points x_0, $x_1 \in X$, there is a path in X from x_0 to x_1.

[*You need to be careful about the use of the word 'any' as it is rather ambiguous. If you write the above condition as 'for any pair of points ... ' it is not really clear whether you mean for each pair of points or for some pair of points. I didn't deduct marks for this usage since is was usually clear from the rest of the answer that those who used it knew the definition but another examiner might not have been so tolerant!*]

[**1 mark, bookwork**]

(b) Saying that path-connectedness is a *topological property* means that, if $X \cong Y$ are homeomorphic topological spaces, then X is path connected if and only if Y is path-connected.

[**1 mark, bookwork**]

(c) To prove this, suppose that X is path-connected. Then, given two points y_0, $y_1 \in Y$ let x_0, $x_1 \in X$ be points such that $f(x_i) = y_i$ (these points exist since f is a bijection). Since X is path-connected there is a path $\sigma\colon I \to X$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. Then $f \circ \sigma\colon I \to Y$ is a path in Y from y_0 to y_1 (since the composition of continuous maps is continuous). Hence Y is path-connected. Conversely, if Y is path-connected then so is X by the same argument (interchanging the roles of X and Y).

[*If you are trying to prove that Y is path-connected then you must start with two points of Y. Quite a lot of people started with points in X. In marking this proof I could tell from the first sentence whether you were going to get it right!*]

[**4 marks, bookwork**]

(d) Given two points x, $x' \in S^1$ we may write $x = (\cos\theta, \sin\theta)$ and $x' = (\cos\theta', \sin\theta')$ for $\theta, \theta' \in \mathbb{R}$. Define $\sigma\colon I \to S^1$ by $\sigma(s) = \big(\cos\big((1-s)\theta + s\theta'\big), \sin\big((1-s)\theta + s\theta'\big)\big)$ for $s \in I$. Then σ is a path in S^1 from x to x'.

[*I was amazed how many people wrote down the straight line between two points writing out a proof that the closed disc D^2 is path-connected for which they got no marks. Other startling answers which gained no marks were claims that S^1 is homeomorphic to the half open interval* $[0, 1)$ *(you can write down a continuous bijection* $[0, 1) \to S^1$ *but it is not a homeomorphism) or the closed interval* $[0, 1]$.]

[**3 marks, exercise set**]
[**Total marks 10**]

A3. (a) X is Hausdorff when for each pair of distinct points x, $y \in X$, there are open subsets U and V in X such that $x \in U$, $y \in V$ and $U \cap V \neq \emptyset$.

[*Again many people used 'for any pair of distinct points ...' with the same ambiguity as in A2(a).*]

[**2 marks, bookwork**]

(b) This space is not Hausdorff because every open subset containing b also contains c and so open subsets as required cannot be found for $x = b$ and $y = c$.

[**2 marks, similar to problems set**]

(c) The subspace topology on $X_1 \subset X$ is given by $\{\, U \cap X_1 \mid U \text{ open in } X \,\}$.

[**2 marks, bookwork**]

(d) Suppose X is a Hausdorff space and x, $y \in X_1 \subset X$ such that $x \neq y$. Then, since X is Hausdorff there are disjoint open subsets U, V in X such that $x \in U$ and $y \in V$. Then $U_1 = U \cap X_1$ is an open subset in X_1 and $x \in U_1$, $V_1 = V \cap X_1$ is an open subset in X_2 and $y \in V_1$. Furthermore $U_1 \cap V_1 \subset U \cap V = \emptyset$ and so U_1 and V_1 are disjoint. Hence X_1 is Hausdorff.

[*This question was on the whole well done. In this sort of question you do need to be careful to specify where subsets are open. If you just say that a subset is 'open' it is not clear whether you mean 'open in X' or 'open in X_1'. It was usually clear from the context what was meant so I didn't deduct marks for lack of clarity about this — but I might have done!*]

[**4 marks, problem set**]
[**Total marks: 10**]

A4. (a) Two paths σ_0 and $\sigma_1\colon I \to X$ from x_0 to x_1 are *homotopic* if there is a continuous map (a homotopy) $H\colon I^2 \to X$ such that

$$\begin{aligned} H(s,0) &= \sigma_0(s), \\ H(s,1) &= \sigma_1(s), \\ H(0,t) &= x_0, \\ H(1,t) &= x_1 \end{aligned}$$

for all s, $t \in I$.

[*It is necessary to say that H is continuous and it is necessary to include the argument (s) in $\sigma_i(s)$ in the first two lines of the conditions.*]

[3 marks, bookwork]

(b) Given two paths σ and τ in X, the product $\sigma * \tau$ is defined when $\sigma(1) = \tau(0)$ and is given by

$$\sigma * \tau(s) = \begin{cases} \sigma(2s) & \text{for } 0 \leqslant s \leqslant 1/2, \\ \tau(2s-1) & \text{for } 1/2 \leqslant s \leqslant 1 \end{cases}$$

for $s \in I$. This function is well-defined by the condition on σ and τ and is continuous by the Gluing Lemma since $[0,1/2]$ and $[1/2,1]$ are closed subsets of I.

[*Again it is necessary to include the argument (s) on the left hand side of the above displayed formula. It is also necessary to indicate why this formula is well-defined and gives a continuous function. Since this is a very basic definition I did expect a reference to the fact that the conditions of the Gluing Lemma were satisfied. I was a bit more relaxed about this in later uses of the Gluing Lemma.*]

[3 marks, bookwork]

(c) Given homotopies of paths $H\colon \sigma_0 \simeq \sigma_1$ and $K\colon \tau_0 \simeq \tau_1$ such that $\sigma_0 * \tau_0$ is defined, then $\sigma_1(1) = \sigma_0(1) = \tau_0(0) = \tau_1(0)$ and so the product $\sigma_1 * \tau_1$ is defined. A homotopy $\sigma_0 * \tau_0 \simeq \sigma_1 * \tau_1$ is given by $H * K\colon I^2 \to X$ where

$$H * K(s,t) = \begin{cases} H(2s,t) & \text{for } 0 \leqslant s \leqslant 1/2, \\ K(2s-1,t) & \text{for } 1/2 \leqslant s \leqslant 1 \end{cases}$$

for s, $t \in I$. This function is well-defined since $H(1,t) = \sigma_0(1) = \tau_0(0) = K(0,t)$ and is continuous by the Gluing Lemma since $[0,1/2] \times I$ and $[1/2,1] \times I$ are closed subsets of I^2.

[*Quite a lot of people defined the homotopy $H * K$ but then failed to go on and say what it was for , namely to prove that $\sigma_0 * \tau_0 \simeq \sigma_1 * \tau_1$.*]

[4 marks, bookwork]
[Total marks 10]

B5. (a) Define an equivalence relation on X by $x \sim x'$ if and only if there is a path in X from x to x'. Then the *path-components* of X are the equivalence classes. **[2 marks, bookwork]**

(b) Suppose that $f\colon X \to Y$ is a continuous map. Then this induces a function $f_*\colon \pi_0(X) \to \pi_0(Y)$ by $f_*[x] = [f(x)]$. This is well-defined because $[x] = [x']$ implies that $x \sim x'$ so that there is a path $\sigma\colon [0,1] \to X$ in X from x to x'. Then $f \circ \sigma\colon [0,1] \to Y$ is a path in Y from $f(x)$ to $f(x')$ and so $f(x) \sim f(x')$, i.e. $[f(x)] = [f(x')]$.

[*A remarkable number of people defined the function $f_*\colon \pi_1(X,x_0) \to \pi_1\big(Y, f(x_0)\big)$ which didn't usually gain any credit.*]

[3 marks, bookwork]

If f is a homeomorphism then f_* is a bijection since the inverse $g = f^{-1}\colon Y \to X$ induces a function $g_*\colon \pi_0(Y) \to \pi_0(X)$ inverse to f_* since $g_*\big(f_*[x]\big) = [g\big(f(x)\big)] = x$ and $f_*\big(g_*[y]\big) = y$.

[2 marks, bookwork]

(c) Suppose that $f\colon X \to Y$ is a homeomorphism and $\{p,q\}$ is a pair of distinct points in X. Then f induces a homeomorphism $X \setminus \{p,q\} \to Y \setminus \{f(p), f(q)\}$ and this induces a bijection $f_*\colon \pi_0(X \setminus \{p,q\}) \to \pi_0(Y \setminus \{f(p), f(q)\}$. Hence $\{p,q\}$ is a cut-pair of type n in X if and only if $\{f(p), f(q)\}$ is a cut-pair of type n in Y.

[*It is essential to say that* $X \setminus \{p,q\} \to Y \setminus \{f(p), f(q)\}$ *is a homeomorphism so that you can use the result of (b). It is not enough to say that it is a bijection.*]

[3 marks, exercise set]

(d) In space (i) there are no cut-pairs of type 3.
In space (ii) there is a unique cut-pair of type 3 (the two points at the ends of the diameter).
In space (iii) there are infinitely many cut-pairs of type 3 (any pair of points on the the radial line (apart from the centre of the circle).
These properties distinguish the three spaces since the cardinality of the number of cut-pairs of type 3 must be the same for homeomorphic spaces by part (c).

[*I was surprised how many people decided to take advantage of 'or otherwise' given that the question is so easy using cut pairs of type 3.*]

[5 marks, unseen]
[Total marks 15]

B6. (a) The *quotient topology* on Y determined by $q\colon X \to Y$ is given by $V \subset Y$ is open in Y if and only if $q^{-1}(V)$ is open in X. **[1 mark, bookwork]**

The *universal property* of the quotient topology is: given a topological space Z, a function $f\colon Y \to Z$ is continuous if and only if $f \circ q\colon X \to Z$ is continuous. **[2 marks, bookwork]**

(b) Given a continuous surjection $f\colon X \to Y$, define an equivalence relation on X by $x \sim x' \Leftrightarrow f(x) = f(x')$. Then we may define $F\colon X/\sim \to Y$ by $F([x]) = f(x)$.
Since $[x] = [x'] \Rightarrow x \sim x' \Rightarrow f(x) = f(x')$ (by the definition of the equivalence relation), the function F is well-defined.
Since $F([x]) = F([x']) \Rightarrow f(x) = f(x') \Rightarrow x \sim x'$ (by the definition of the equivalence relation) $\Rightarrow [x] = [x']$, F is a monomorphism.
Since f is a surjection, $y = f(x)$ for some $x \in X$ and so $y = F([x])$. Hence F is a surjection.
This shows that $F\colon X/\sim \to Y$ is a bijection. **[4 marks, bookwork]**

The function $F\colon X/\sim \to Y$ is continuous by the universal property since $F \circ q = f$ which is given as continuous. **[1 mark, bookwork]**

The space $X/\sim = q(X)$ is compact since it is the continuous image of a compact set. Hence F is a homeomorphism since it is a continuous bijection from a compact space to a Hausdorff space. **[2 marks, bookwork]**

(c) Define $f\colon X \to D^2$ by $f(x) = (|x| - 1)x/|x|$ This is a continuous surjection. Furthermore $f(x) = f(x')$ if and only if $x = x'$ or $|x| = |x'| = 1$, i.e. $x = x'$ or $x,\, x' \in S^1$. Thus the general result in (b) gives the required homeomophism $F\colon X/S^1 \to D^2$ since X is compact (a closed bounded subset of $\mathbb{R}^2$) and D^2 is Hausdorff (a subspace of $\mathbb{R}^2$).

[*Here it is necessary to find a continuous surjection* $f\colon X \to D^2$ *so that* $f(x) = f(x') \Leftrightarrow x = x'$ *or* $x, x' \in S^1$. *You can then apply the result of (b) once you have said why* X *is compact and* D^2 *is Hausdorff. Saying this gained three marks. Quit a lot of people tried the function given by* $f(x) = (|x| - 1)x$ *which does satisfy the above condition but gives a map from* X *onto the* *closed disc of radius 2 rather than the disc of radius 1.*]

[**5 marks, similar to exercises set**]
[**Total marks 15**]

B7. (a) The *fundamental group* of X based at x_0, $\pi_1(X, x_0)$ is the set of homotopy classes of loops in X based at x_0 (i.e. paths in X from x_0 to x_0) with multiplication given by $[\sigma][\tau] = [\sigma * \tau]$.

The group product is well-defined since, if $\sigma_0 \simeq \sigma_1$ and $\tau_0 \simeq \tau_1$ are pairs of homotopic loops based at x_0 then $\sigma_0 * \tau_0 \simeq \sigma_1 * \tau_1$.

The group product is associative since $(\sigma * \tau) * \rho \simeq \sigma * (\tau * \rho)$ for loops σ, τ and ρ based at x_0.

The group identity is $[\varepsilon_{x_0}]$, the class of the constant loop, since $\varepsilon_{x_0} * \sigma \simeq \sigma \simeq \sigma * \varepsilon_{x_0}$.

The inverse of $[\sigma]$ is given by the homotopy class of the reverse loop $[\overline{\sigma}]$ since $\sigma * \overline{\sigma} \simeq \varepsilon_{x_0} \simeq \overline{\sigma} * \sigma$.

[**7 marks, bookwork, some of which has to be summarized**]

(b) A topological space X is *simply-connected* if it is path-connected and if the fundamental group $\pi_1(X, x_0)$ for some base point $x_0 \in X$ is a trivial group.

[*Quite a lot of people forgot to include 'path-connected' in their definition.*]

[**2 marks, bookwork**]

(c) Suppose that that all paths from x_0 to x_1 are equivalent. Let τ be a path from x_0 to x_1. Then given a loop σ at x_0, $\sigma * \tau$ is a path from x_0 to x_1 and so $\sigma * \tau \simeq \tau$ or, equivalently, $[\sigma][\tau] = [\tau]$. Hence, $[\sigma][\tau][\overline{\tau}] = [\tau][\overline{\tau}]$. But $[\sigma][\tau][\overline{\tau}] = [\sigma][\varepsilon_{x_0}] = [\sigma]$ and $[\tau][\overline{\tau}] \simeq [\varepsilon_{x_0}]$ so that $[\sigma] = [\varepsilon_0]$. Hence $\pi_1(X, x_0) = \{[\varepsilon_{x_0}]\}$, the trivial group, and so X is simply connected.

[*Quite a lot of people misread this question. You are given the points x_0 and x_1 and so cannot choose them how you like. In particular you cannot suppose that $x_1 = x_0$ which does make the result of part (c) rather easy since it would say that all loops based at x_0 are homotopic, and so in particular are homotopic to the constant loop. This argument gained no credit. Another argument which did gain a mark was to observe that, if σ and τ are two paths from x_0 to x_1 then $\sigma * \overline{\tau}$ is a loop based at x_0 which is homotopic to the constant loop (since $\sigma * \overline{\tau} \simeq \sigma * \overline{\sigma} \simeq \varepsilon_{x_0}$). The problem is that most loops based at x_0 are not of this form.*]

[**6 marks, exercise set**]
[**Total marks 15**]

B8. (a) Suppose that $\sigma\colon I \to S^1$ is a loop in S^1 based at 1. Then by the Path-Lifting Theorem, there is a unique path $\widetilde{\sigma}\colon I \to \mathbb{R}$ such that $p \circ \widetilde{\sigma} = \sigma$ and $\widetilde{\sigma}(0) = 0$. Since $p \circ \widetilde{\sigma}(1) = \sigma(1) = 1$, $\widetilde{\sigma}(1) \in \mathbb{Z}$. We define this integer to be the *degree* of the loop σ, written $\deg(\sigma)$.

[*When you refer to the lift $\widetilde{\sigma}$ is necessary to say that this is a continuous map to $\mathbb{R}$.*]

[**3 marks, bookwork**]

(b) By the Monodromy Theorem, if $\sigma_0 \simeq \sigma_1$ are homotopic loops in S^1 based at 1 with lifts $\widetilde{\sigma}_0$ and $\widetilde{\sigma}_1$ as above, then $\widetilde{\sigma}_0(1) = \widetilde{\sigma}_1(1)$ and so the function $\phi\colon \pi_1(X, x_0) \to \mathbb{Z}$ given by $\phi([\sigma]) = \deg(\sigma)$ is well-defined. [**2 marks, bookwork**]

To see that ϕ is a homomorphism, let σ and τ be loops in S^1 based at 1 with lifts $\widetilde{\sigma}$ and $\widetilde{\tau}$ as above. Then we may define a lift for the loop $\sigma * \tau$ by

$$\rho(s) = \begin{cases} \widetilde{\sigma}(2s) & \text{for } 0 \leqslant s \leqslant 1/2, \\ \widetilde{\sigma}(1) + \widetilde{\tau}(2s-1) & \text{for } 1/2 \leqslant s \leqslant 1. \end{cases}$$

which is well-defined and continuous by the Gluing Lemma. Hence $\deg(\sigma * \tau) = \rho(1) = \widetilde{\sigma}(1) + \widetilde{\tau}(1)$ and so $\phi([\sigma][\tau]) = \phi([\sigma]) + \phi([\tau])$. [**4 marks, bookwork**]

ϕ is an epimorphism since, given $n \in \mathbb{Z}$, the path $\widetilde{\sigma}$ in $\mathbb{R}$ given by $\widetilde{\sigma}(s) = ns$ projects to a loop $\sigma = p \circ \widetilde{\sigma}$, $\sigma(s) = \exp(2\pi i n s)$ of degree n since $\widetilde{\sigma}(1) = n$. **[2 marks, bookwork]**

(c) Consider the loop in S^1 based at 1, $\sigma(s) = \exp(2\pi i n s)$. Then (as above) $\deg(\sigma) = n$ and so $\phi(\sigma) = n$. For $f \colon S^1 \to S^1$ given by $f(z) = \overline{z}$, $f_*([\sigma])$ is represented by the loop $f \circ \sigma$. However, $f \circ \sigma(s) = \exp(-2\pi i n s)$ whose lift as above is given by $\widetilde{f \circ \sigma}(s) = -ns$ so that $\deg(f \circ \sigma) = -n$. Thus the homomorphism $f_* \colon \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ corresponds to $n \mapsto -n$.

[4 marks, similar to exercise set]

[Total marks 15]

C9. (a) $N \subset X$ is a *neighbourhood* of $x_0 \in N$ when there is an open set U such that $x_0 \in U \subset N$.

Suppose that $U \subset X$ is open. Then, for $x_0 \in U$, $x_0 \in U \subset U$ and so U is a neighbourhood of x_0.

Conversely, if U is a neighbourhood of each of its points, then, for each $x \in U$, there is an open set U_x such that $x \in U_x \subset U$. Then $U = \bigcup_{x \in U} U_x$ is a union of open sets and so is open. **[5 marks, exercise set]**

(b) $f \colon X \to Y$ is continuous at $x_0 \in X$ when, for each neighbourhood N of $f(x_0)$ in Y, $f^{-1}(N)$ is a neighbourhood of x_0 in X.

Suppose that $f \colon X \to Y$ is a continuous function of topological spaces. Given $x \in X$ suppose that N is a neighbourhood of $f(x)$ in Y. Then, by definition, there is an open subset U of Y such that $f(x) \in U \subset N$. It follows that $x \in f^{-1}(U) \subset f^{-1}(N)$ and so, since $f^{-1}(U)$ is open in X (because f is continuous), $f^{-1}(N)$ is a neighbourhood of x as required to prove that f is continuous at x.

Conversely, suppose that $f \colon X \to Y$ is continuous at each $x \in X$ and suppose that V is an open subset of Y. Suppose that $x \in f^{-1}(V)$. Then $f(x) \in V$ and so V is a neighbourhood of $f(x)$ since V is open. Hence $f^{-1}(V)$ is a neighbourhood of x since f is continuous at x. Hence, $f^{-1}(V)$ is a neighbourhood of each of its points and so, by part (a), is open as required to prove that f is continuous. **[5 marks, exercise set]**

(c) A point $x \in X$ is a interior point of A if A is a neighbourhood of x. The set of interior points of A is called the *interior* of A and is denoted A°.

Suppose that A is open. Then, for $x \in A$, by (a) A is a neighbourhood of x and so $x \in A^\circ$. Hence $A \subset A^\circ$. But, for $x \in A^\circ$, $x \in U \subset A$ for some open set U and so $x \in A$. Hence $A^\circ \subset A$ so that $A^\circ = A$.

Conversely, if $A^\circ = A$, then each $x \in A$ is an interior point of A and so A is a neighbourhood of each of its points and so is open by part (a). **[5 marks, exercise set]**

(d) (i) For each $x \in (0,1)$, $x \in [x,1) \subset (0,1)$ and so $(0,1)$ is a neighbourhood of x. Hence $(0,1)^\circ = (0,1)$.

(ii) $[0,1)$ is open and so $[0,1)^\circ = [0,1)$.

(iii) By (i) and (c), $(0,1)$ is open and so $(0,1) = (0,1)^\circ \subset (0,1]^\circ$ so that $(0,1) \subset (0,1)^\circ$. However, $(0,1]$ is not a neighbourhood of 1 since the basic open subsets containing 1 all have the form $[a,b)$ where $a \leqslant 1 < b$ which includes $1 + b/2 \notin (0,1]$. So any open subset containing 1 also contains a point not in $(0,1]$. Hence $(0,1]^\circ = (0,1)$.

(iv) $[0,1)$ is open and so $[0,1) = [0,1)^\circ \subset [0,1]^\circ$ so that $[0,1) \subset [0,1]^\circ$. By a similar argument to (iii), $1 \notin [0,1]^\circ$. Hence, $[0,1]^\circ = [0,1)$. **[5 marks, unseen]**

[Total marks 20]

C10. (a) Given a group G and a set X, then a *G-action* on the set X is a function $G \times X \to X$, written $(g, x) \mapsto g \cdot x$, such that $1 \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all g, $h \in G$ and $x \in X$ (where the group is written multiplicatively with identity 1). A topological space X is a G-space if there is a G-action $G \times X \to X$ such that the function $\theta_g \colon X \to X$ defined by $\theta_g(x) = g \cdot x$ is continuous for all $g \in G$.

For g, $h \in G$, $\theta_{gh}(x) = (gh) \cdot x = g \cdot (h \cdot x) = \theta_g\big(\theta_h(x)\big)$ and so $\theta_{gh} = \theta_g \circ \theta_h$. Also $\theta_1(x) = 1 \cdot x = x$ and so $\theta_1 = 1_X$, the identity function. Hence the function $\theta_g \colon X \to X$ is a homeomorphism since $\theta_{g^{-1}}$ is its inverse and so the function $g \mapsto \theta_g$ gives a homomorphism from G to the group of homeomorphisms $X \to X$. [**5 marks, similar to exercise set**]

(b) A topological space X is *normal* if, for each pair of disjoint closed sets A and B in X, there exists disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.

Suppose that A and B are disjoint non-empty closed sets in a compact Hausdorff space X (the result is trivial if either A or B is empty). Let $a \in A$, then, for each $b \in B$ we can find disjoint open sets U_b and V_b such that $a \in U_b$ and $b \in V_b$. The set $\{\, V_b \mid b \in B \,\}$ is an open cover for B. However B is a closed set in a compact space and so is compact (by Proposition 5.7). Hence there is a finite subcover $\{\, V_{b_1}, \ldots, V_{b_n} \,\}$ for B. Put $W_a = U_{b_1} \cap \ldots \cap U_{b_n}$ and $W'_a = V_{b_1} \cup \ldots \cup V_{b_n}$. Then, by the choice of the points b_i, $B \subset W'_a$ and, since $a \in U_b$ for all $b \in B$, $a \in W_a$. Finally, since $U_b \cap V_b = \emptyset$, $W_a \cap V_b = \emptyset$ and so $W_a \cap W'_a = \emptyset$.

Now $\{W_a \mid a \in A\}$ is on open cover for A. Since A is closed in compact X then A is compact. Hence there is a finite subcover $\{W_{a_1}, \ldots, W_{a_m}\}$ for A. Put $U = W_{a_1} \cup \ldots \cup W_{a_m}$ and $V = W'_{a_1} \cap \ldots \cap W'_{a_m}$. Then, as above, $A \subset U$ and $B \subset V$ and $U \cap V = \emptyset$ as required.

[**10 marks, exercise set**]
[**Total marks 15**]

C11. (a) A continuous map $p \colon \widetilde{X} \to X$ is a *covering* if (i) p is a surjection; (ii) for each $x \in X$ there is an open neighbourhood V of x such that $p^{-1}(V) = \bigcup_{\lambda \in \Lambda} U_\lambda$, a disjoint union of open subsets of $\widetilde{X}$ such that the restriction map $p|U_\lambda \colon U_\lambda \to V$ is a homeomorphism for each $\lambda \in \Lambda$.

[**4 marks, bookwork**]

(b) Suppose that X is a G-space for a group G. Then the action of G on X is said to be *properly discontinuous* if, for each $x \in X$, there is an open neighbourhood U of x such that $\theta_{g_1}(U) \cap \theta_{g_2}(U) = \emptyset$ for all $g_1 \neq g_2 \in G$. [**2 marks, bookwork**]

(c) For $x \in \mathbb{R}$ we may take $U = (x - 1/2, x + 1/2)$ since for $n_1 \neq n_2 \in \mathbb{Z}$, $\theta_{n_i}(U) = (x + n_i - 1/2, x + n_i + 1/2)$. Then $\theta_{n_1}(U) \cap \theta_{n_2}(U) = \emptyset$ since if $n_1 < n_2$ (without loss of generality) $x + n_1 + 1/2 \leqslant x + n_2 - 1/2$ since $n_2 - n_1 \geqslant 1$.

[**3 marks, bookwork with details omitted**]

(d) The quotient map $q \colon X \to X/G$ is continuous and a surjection by definition. Furthermore q is an open map (given). Given $x \in X$, let U be an open neighbourhood of x as given by proper discontinuity. Then $V = q(U)$ an open neighbourhood of $[x] \in X/G$ and $q^{-1}(V) = \bigcup_{g \in G} \theta_g(U)$, a disjoint union. Finally, the restriction of q gives a continuous open bijection $\theta_g(U) \to V$ and so a homeomorphism. Hence q is a covering. [**6 marks, exercise set**]

[**Total marks 15**]