Solutions 3

1. Suppose that a subset $U \subset X_2$ is open in the subspace topology on $X_2$ induced by the subspace topology on $X_1$. Then $U = V \cap X_2$ where $V$ is open in the subspace topology on $X_1$. Since $V$ is open in the subspace topology on $X_1$, $V = W \cap X_1$ where $W$ is open in $X$. But then $U = V \cap X_2 = (W \cap X_1) \cap X_2 = W \cap X_2$ and so $U$ is open in the subspace topology on $X_2$ induced by the topology on $X$.

Conversely, if $U$ is open in the subspace topology on $X_2$ induced by $X$, then $U = W \cap X_2$ where $W$ is open in $X$. But then $U = (W \cap X_1) \cap X_2$. Here $W \cap X_1$ is open in the subspace topology on $X_1$ and so $U$ is open in the subspace topology on $X_2$ induced by the subspace topology on $X_1$.

2. Suppose that $V \subset X$ is an open subset in the subspace topology. Then $V = U \cap X$ where $U \subset \mathbb{R}^n$ is an open subset in $\mathbb{R}^n$ with respect to the usual topology on $\mathbb{R}^n$. To prove that $V$ is open in $X$ in the usual topology let $x_0 \in V$. Then we must prove that $V$ is a neighbourhood of $x_0$ in $X$. Since $V \subset U$, $x_0 \in U$ and so, since $U$ is an open subset of $\mathbb{R}^n$, $U$ is a neighbourhood of $x_0$ in $\mathbb{R}^n$ in the usual topology, i.e. there exists $\varepsilon > 0$ so that $B_\varepsilon(x_0) \subset U$. Hence $B_\varepsilon^X(x_0) = B_\varepsilon(x_0) \cap X \subset U \cap X = V$ so that $V$ is a neighbourhood of $x_0$ in $X$ as required.

Conversely, suppose that $V \subset X$ is an open subset in the usual topology. Then, for each $x \in V$, there is a real number $\varepsilon_x > 0$ such that $B_\varepsilon^X(x) \subset V$. Then $V = \bigcup_{x \in V} B_\varepsilon^X(x) = \bigcup_{x \in V} (B_\varepsilon(x) \cap X) = (\bigcup_{x \in V} B_\varepsilon(x)) \cap X$. Hence $U = \bigcup_{x \in V} B_\varepsilon(x)$ is an open subset of $\mathbb{R}^n$ in the usual topology so that $V = U \cap X$. Hence $U$ is open in $X$ in the subspace topology.

3. Suppose that $A_1$ is a closed subset of $X_1$. Then $X_1 \setminus A_1$ is an open subset of $X$. Hence, $X_1 \setminus A_1 = U \cap X_1$ where $U$ is an open subset of $X$. Then $A_1 = (X \setminus U) \cap X_1 = A \cap X_1$ where $A$ is closed in $X$ as required.

Conversely, if $A_1 = A \cap X_1$ where $A$ is closed in $X$, then $X_1 \setminus A_1 = (X \setminus A) \cap X_1$ which is therefore open in $X_1$ and so $A_1$ is closed in $X_1$.

4. The argument here is identical to the proofs of Proposition 3.6 and Theorem 3.7 with ‘closed’ replaced by ‘open’ throughout.
5. Let $X_1 = [0, \infty)$ and $X_2 = (-\infty, 0)$ with usual topology. Then $X_1 \cup X_2 = \mathbb{R}$. The constant function $c_0: X_1 \to \mathbb{R}$ is continuous and the constant function $c_1: X_2 \to \mathbb{R}$ is continuous (since constant functions are always continuous) but the glued up function is not.

6. The homeomorphism is given by
   \[
   f: \mathbb{R} \times S^1 \to \mathbb{R}^2 \setminus \{0\}; \quad (s, x, y) \mapsto e^s \cdot (x, y)
   \]
   with inverse
   \[
   f: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \times S^1; \quad (x, y) \mapsto \left(\ln(|(x, y)|), \frac{(x, y)}{|(x, y)|}\right).
   \]

8. For $x_0 = (t_0, i)$ and $x_1 = (t_1, j)$ we can use the path $\sigma: [0, 1] \to X$ with
   \[
   \sigma(s) = \begin{cases} 
   ((1-s)t_0 + st_1, i) & s \neq 1, \\
   ((1-s)t_0 + st_1, j) & s = 1.
   \end{cases}
   \]
   this is map is continuous, since the composition $p_1 \circ \sigma: [0, 1] \to \mathbb{R}$ with $(p_1 \circ \sigma)(s) = (1-s)t_0 + st_1$ is a continuous map and the composition with $p_2 \circ \sigma: \{0, 1\}$ is a map to an indiscrete space and therefore continuous as well. Now the universal property of the product topology implies continuity of $\sigma$.

9. We may check the conditions for a topology as follows.
   (i) $\emptyset \cap X_i = \emptyset$ which is open in $X_i$ (for $i = 1, 2$) and so $\emptyset$ is open in $X_1 \cup X_2$. $(X_1 \cup X_2) \cap X_i = X_i$ which is open in $X_i$ (for $i = 1, 2$) and so $X_1 \cup X_2$ is open in $X_1 \cup X_2$.
   (ii) Suppose that $U_1$ and $U_2 \subset X_1 \cup X_2$ are open in $X_1 \cup X_2$. Then $U_1 \cap X_j$ is open in $X_j$ for $j = 1, 2$. Hence $(U_1 \cap U_2) \cap X_j = (U_1 \cap X_j) \cap (U_2 \cap X_j)$ is open in $X_j$ for $j = 1, 2$ and so $U_1 \cap U_2$ is open in $X_1 \cup X_2$.
   (iii) Suppose that $U_\lambda$ is open in $X_1 \cup X_2$ for $\lambda \in \Lambda$. Then $\bigcup_{\lambda \in \Lambda} U_\lambda \cap X_j = \bigcup_{\lambda \in \Lambda} (U_\lambda \cap X_j)$ which is open in $X_j$ and so $\bigcup_{\lambda \in \Lambda} U_\lambda$ is open in $X_1 \cup X_2$ as required.
   Hence the definition in the question does give a topology on $X_1 \cup X_2$.
   The inclusion map $i_j: X_j \to X_1 \cup X_2$ is continuous since, given an open subset $U \subset X_1 \cup X_2$, $i_j^{-1}(U) = U \cap X_j$ which is open in $X_j$ by definition. Hence, if $f: X_1 \cup X_2 \to Y$ is continuous, the restricted functions $f|_{X_j} = f \circ i_j: X_j \to Y$ is continuous since the composition of continuous functions is continuous.
Conversely, suppose that $f: X_1 \sqcup X_2 \to Y$ is such that the restrictions $f_j = f|X_j$ are continuous. Then, given an open set $V \subset Y$, the set $f_j^{-1}(V)$ is open in $X_j$. Hence $f^{-1}(V)$ is open in $X_1 \sqcup X_2$ since $f^{-1}(V) \cap X_j = f_j^{-1}(V)$. Hence $f$ is continuous.