1. In the indiscrete topology on $X$ there are only two open sets ($\emptyset$ and $X$) and so any open cover of any subset $A$ is already finite. Hence $A$ is compact.

2. Suppose that $K_1$ and $K_2$ are compact subsets and $\mathcal{F}$ is an open cover for $K_1 \cup K_2$. Then $\mathcal{F}$ is an open cover for $K_1$ and so has a finite subcover $\mathcal{F}_1$ for $K_1$ and similarly $\mathcal{F}$ has a finite subcover $\mathcal{F}_2$ for $K_2$. Then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a finite subcover for $K_1 \cup K_2$. Hence $K_1 \cup K_2$ is compact.

Now suppose that $K_i$ are compact subsets for $1 \leq i \leq n$. We prove that $\bigcup_{i=1}^n K_i$ is compact by induction on $n$. The result is trivial for $n = 1$. Suppose as inductive hypothesis that the result is true for $n = k$. The it follows that it true for $n = k + 1$ by the above argument since $\bigcup_{i=1}^{k+1} K_i = (\bigcup_{i=1}^k K_i) \cup K_{k+1}$ proving the inductive step. Hence the result is true for all $n$.

For a counterexample, the singleton subset $\{n\}$ in $\mathbb{R}$ with the usual topology is compact but the union $\bigcup_{n \in \mathbb{Z}} \{n\} = \mathbb{Z}$ is not compact.

3. Suppose that $\mathcal{F}$ is an open cover for a non-empty open subset $A \subset \mathbb{R}$ in the cofinite topology. For $a \in A$, $a \in U_0$ for some open subset $U_0$ in $\mathcal{F}$. Then either $U_0 = \mathbb{R}$, in which case $A \subseteq U_0$ so that $\{U_0\}$ is a finite subcover for $A$, or $U_0 = \mathbb{R} \setminus \{x_1, x_2, \ldots, x_n\}$ the complement of some finite set $\{x_1, x_2, \ldots, x_n\}$. Reordering this finite set if necessary, suppose that $\{x_1, \ldots, x_k\}$ are the points of the finite set which lie in $A$. For each such point, since $\mathcal{F}$ is an open cover for $A$ there must be some open subset $U_i \in \mathcal{F}$ such that $x_i \in U_i$. Then $\{U_i \mid 0 \leq i \leq k\}$ is a finite subcover for $A$ as required to prove that $\mathbb{R}$ with this topology is compact.

4. (a) This is immediate since each open subset in this topology is open in the usual topology.

(b) Suppose that $\mathcal{F}$ is an open cover for $[a, b)$. Then there must be an open set $U \in \mathcal{F}$ such that $a \in U$. But, by the definition of the topology, $U = (a_1, \infty)$ for some $a_1 \in \mathbb{R}$. Since $a \in (a_1, \infty)$ we must have $a_1 < a$ and so $[a, b) \subseteq (a_1, \infty)$ so that $\{U\}$ is a finite subcover for $[a, b)$ consisting of a single open subset. Hence $[a, b)$ is compact. Similarly, $[a, \infty)$ is compact.

(c) The open covering $\{ (a + 1/n, \infty) \mid n \geq 1 \}$ for $[a, b]$ has no finite subcover.
so \((a, b]\) is not compact.
The open covering \(\{(-n, \infty) \mid n \geq 1\}\) for \((-\infty, b]\) has no finite subcover and so \((-\infty, b]\) is not compact.

5. Suppose for contradiction that the subsets \(A_n\) are as in the question but that \(\bigcap_{n=1}^{\infty} A_n = \emptyset\). Then \(\bigcup_{n=1}^{\infty} (X \setminus A_n) = X \setminus \bigcap_{n=1}^{\infty} A_n = X\) and so \(\{X \setminus A_n \mid n \geq 1\}\) is an open cover for \(X\) and so for \(A_1\). But now, since \(A_1\) is compact, there must be a finite subcover of \(A_1\). Now notice that the nesting of the subsets \(A_n\) means that \(X \setminus A_k \subseteq X \setminus A_{k+1} \subseteq \cdots \subseteq X \setminus A_1 \subseteq X\) and so \(\{X \setminus A_n \mid n \geq 1\}\) is an open cover for \(X\) and so for \(A_1\). But now, since \(A_1\) is compact, there must be a finite subcover of \(A_1\). Now notice that the nesting of the subsets \(A_n\) means that \(X \setminus A_1 \subseteq X \setminus A_2 \subseteq \cdots \subseteq X \setminus A_n \subseteq X \setminus A_{n+1} \subseteq \cdots\).

Let \(X \setminus A_k\) be the largest subset in the finite subcover. Then \(A_1 \subseteq X \setminus A_k\) (since \(X \setminus A_n \subseteq X \setminus A_k\) for \(n \leq k\) by the nesting of the subsets \(A_n\)). Hence \(A_k \subseteq A_1 \subseteq X \setminus A_k\) which implies that \(A_k = A_k \cap A_1 = \emptyset\) contradicting the choice of the sets \(A_n\) as non-empty subsets. Hence, \(\bigcap_{n=1}^{\infty} A_n\) is non-empty as required.

[This result is important in dynamical systems.]

6. Suppose that \(A\) is a closed subset of a compact Hausdorff space \(X\) and \(b\) is a point of \(X\) such that \(b \notin A\). Since \(X\) is a Hausdorff space, for each point \(a \in A\) there are disjoint open subsets \(U_a\) and \(V_a\) such that \(a \in U_a\) and \(b \in V_a\). Then the collection of open subsets \(\{U_a \mid a \in A\}\) is an open cover for \(A\) since each point of \(A\) lies in \(U_a\), one of the open subsets in the covering. Now, since \(X\) is compact and \(A\) is a closed subset, \(A\) is compact (Proposition 4.6). Hence there is a finite subcover \(\{U_{a_i} \mid 1 \leq i \leq n\}\) for \(A\). This means that \(A \subseteq \bigcup_{i=1}^{n} U_{a_i} = U\), say. \(U\) is a union of open subsets and so is an open subset. Now let \(V = \bigcap_{i=1}^{n} V_{a_i}\). Then \(V\) is a finite intersection of open subsets and so is open. By definition \(b \in V_a\) for all \(a\) and so \(b \in V\).

Finally, \(U\) and \(V\) are disjoint. To see this, observe that, for \(1 \leq i \leq n\), \(U_{a_i} \cap V_{a_i} = \emptyset\) and so \(U_{a_i} \cap V = \emptyset\) since \(V \subseteq V_{a_i}\). Hence \(U \cap V = \emptyset\) by the definition of \(U\). Hence \(U\) and \(V\) are disjoint open subsets as required.

[Notice that we needed the compactness of \(A\) in order to get a finite intersection. Only a finite intersection of open subsets can be guaranteed to be open. This style of proof is one of the most important applications of compactness.]

7. (a) Problems 4, Question 2. \(X\) is compact because it is a closed bounded set in \(\mathbb{R}^2\) with the usual topology and so \(X/S^1 = q(X)\) is compact since is the continuous image of a compact set. \(D^2\) is Hausdorff since it is a subspace of \(\mathbb{R}^2\) with the usual topology. Hence the function \(F: X/S^1 \to D^2\)
is a continuous bijection from a compact space to a Hausdorff space and is therefore a homeomorphism.

(b) Problems 4, Question 3. \( S^1 \subset \mathbb{R}^2 \) and \([-1, 1] \subset \mathbb{R}\) are compact as closed bounded sets in Euclidean spaces with the usual topology and so \( S^1 \times [-1, 1] \) is compact as the product of two compact spaces. Hence the identification space in the question is compact since it is the continuous image of a compact set. \( S^2 \) is Hausdorff since it is a subspace of \( \mathbb{R}^3 \) with the usual topology. Hence the function \( F \) in the solution is a continuous bijection from a compact space to a Hausdorff space and is therefore a homeomorphism.

(c) Problems 4, Question 4. \( S^1 \) and \([-1, 1]\) are compact (both closed bounded sets in Euclidean spaces with the usual topology) and so the product \( S^1 \times [-1, 1] \) is compact. Hence \( S^1 \times [-1, 1]/\sim = q(S^1 \times [-1, 1]) \) is compact. \( S^1 \) is Hausdorff (subspace of Euclidean space) and so \( S^1 \times S^1 \) is Hausdorff. Hence the function \( F \) in the solution is a continuous bijection from a compact space to a Hausdorff space and is therefore a homeomorphism.

(d) Problems 4, Question 6. \( P^2 \) is the continuous image of compact \( S^2 \) and so is compact. \( F(P^2) \) is a subspace of \( \mathbb{R}^4 \) and so is Hausdorff. Hence the continuous bijection \( F: P^2 \to F(P^2) \) is a homeomorphism.

(e) Problems 4, Question 8. \( D^2 \subset \mathbb{R}^2 \) is compact as a closed bounded subset of a Euclidean space with the usual topology and so the identification space in the question is compact since it is the continuous image of a compact set. \( P^2 \) is Hausdorff because, by Problems 4, Question 6 and Problems 7, Question 7(d), it is homeomorphic to a subspace of \( \mathbb{R}^4 \) with the usual topology. Hence the continuous function \( F \) in the solution is a continuous bijection from a compact space to a Hausdorff space is a homomorphism.
8. (a) We prove that \( b \in K \) by contradiction. Suppose for contradiction that \( b \notin K \) so that \( b \in \mathbb{R} \setminus K \), an open set. Then by the definition of the open sets in the usual topology there is a real number \( \varepsilon > 0 \) such that \( (b - \varepsilon, b + \varepsilon) \subseteq \mathbb{R} \setminus K \) so that \( b - \varepsilon \) is an upper bound for \( K \), contradicting the definition of \( b \) as the least upper bound. Hence \( b \in K \).

(b) By a similar argument, if \( a = \inf K \) (the greatest lower bound) then \( a \in K \). Hence \( K \subseteq [a, b] \), \( a \in K \) and \( b \in K \). However, since \( K \) is a path-connected there is a path in \( K \) from \( a \) to \( b \) and by the Intermediate Value Theorem (Theorem 0.23) every point of \([a, b]\) lies on this path. Hence \([a, b] \subseteq K\).

(c) If \( f : X \to \mathbb{R} \) is a continuous function from a non-empty path-connected compact space \( X \) then \( f(X) \) is a non-empty path-connected compact subset of \( \mathbb{R} \) since the continuous image of a path-connected space is path connected (Problems 1, Question 5) and the continuous image of a compact set is compact (Proposition 4.3). Hence \( f(X) \) is a closed interval \([a, b]\) by part (b).

9. In the example given, \( I_0 = (0, 1) \) which divides into two subintervals \((0, 1/2]\) and \([1/2, 1)\). The second of the subintervals lies in \((1/3, 1)\) and so there is a finite subcover for this subinterval. However, there is not for \((0, 1/2]\) and so \( a_1 = 0 \) and \( b_1 = 1/2 \). This interval divides into two subintervals \((0, 1/4]\) and \([1/4, 1/2)\). The second of these lies in \((1/5, 1)\) but there is no subcover for the first. Hence \( a_2 = 0 \) and \( b_2 = 1/4 \). Continuing in this way we find that \( a_n = 0 \) for all \( n \) and \( b_n = 1/2^n \) so that \( \alpha = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \alpha = 0 \). At this point the proof breaks down because \( \alpha = 0 \notin (0, 1) \) and \( \alpha = 0 \) does not lie in any open set of the cover.

10. First check that \( \tau^* \) is a topology. Indeed \( X^* = X \setminus \emptyset \cup \{\infty\} \in \tau^* \) and \( \emptyset \in \tau \subseteq \tau^* \). Moreover, if \( U, V \in \tau^* \) then \( U \cap V \in \tau^* \) this is clear if both are in \( \tau \). Assume \( U \in \tau \) and \( V = X \setminus C \cup \{\infty\} \) then \( U \cap V = U \cap (X \setminus C) \in \tau \subseteq \tau^* \), since \( C \) is closed in \( X \) by the Hausdorff property. Assume \( U = X \setminus C \cup \{\infty\} \) and \( V = X \setminus K \cup \{\infty\} \) then \( U \cap V = X \setminus (C \cup K) \cup \{\infty\} \in \tau^* \), since a finite union of compact sets is compact by Problem 6.2.
Now, consider a union of open sets in $\tau^*$.

$$
\bigcup_{\lambda} U_\lambda \cup \bigcup_{\mu} V_\mu.
$$

with $U_\lambda \in \tau$ and

$$
V_\mu = X \setminus C_\mu \cup \{\infty\}.
$$

Now, $U = \bigcup_\lambda U_\lambda \in \tau \subset \tau^*$ since $\tau$ is a topology and

$$
V = \bigcup_{\mu} V_\mu = X \setminus \bigcap_{\mu} C_\mu \cup \{\infty\} \in \tau^*;
$$

since $\bigcap_{\mu} C_\mu$ is a closed subset of a compact set (of every $C_\mu$), hence, it is compact. Now it remains to show that $U \cup V$ for $U \in \tau$ and $V = X \setminus C \cup \{\infty\}$ is open:

$$
U \cup V = X \setminus (C \cap (X \setminus U)) \cup \{\infty\}.
$$

But $(X \setminus U)$ is closed in $X$ hence $(C \cap (X \setminus U)) \subset C$ is a closed subset of a compact set. Hence, it is compact and $X \setminus (C \cap (X \setminus U)) \cup \{\infty\} \in \tau^*$.

Now, consider an open cover $\mathcal{F}$ of $X^*$. In order to cover $\infty$ it has to include at least one open subset $U_\infty$ of the form $X \setminus C \cup \{\infty\}$ where $C \subset X$ is compact. Now, $\mathcal{F}' = \{U \cap X \mid U \in \mathcal{F}\}$ is an open cover of $X$ (since $U$ and $X$ are open in $X^*$) and hence of $C$.

By compactness of $C$ a finite subcover $\{U_1 \cap X, \ldots, U_m \cap X\} \subset \mathcal{F}$ suffices to cover $C$. But then one has the finite subcover $\{U_\infty, U_1, \ldots, U_m\} \subset \mathcal{F}$. 

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