1. (a) We define a homotopy $H : I^2 \to X$ by

$$H(s, t) = \begin{cases} 
\sigma \left( \frac{s}{(1+t)/2} \right) & \text{for } 0 \leq s \leq (1+t)/2, \\
\sigma_1 & \text{for } (1+t)/2 \leq s \leq 1.
\end{cases}$$

This is well-defined and is continuous by the Gluing Lemma. It gives the required homotopy $H : \sigma * \varepsilon_{x_0} \sim \sigma$.

(b) Suppose that $H : \sigma_0 \sim \sigma_1$. Define $\overline{H} : I^2 \to X$ by $\overline{H}(s, t) = H(1-s, t)$. Then $\overline{H} : \sigma_0 \sim \sigma_1$.

2. Given two paths $\sigma_0$ and $\sigma_1$ from $x_0$ to $x_1$ in $X$ we may define a homotopy $H: \sigma_0 \sim \sigma_1$ by $H(s, t) = (1-t)\sigma_0(s) + t\sigma_1(s) \in X$ (since $X$ is convex). In particular, given a loop $\sigma$ based at $x_0$ then $\sigma \sim \varepsilon_{x_0}$ and so $\pi_1(X, x_0) = \{[\varepsilon_{x_0}]\} \cong I$, the trivial group.

3. ‘$\Rightarrow$’ Suppose that $X$ is simply connected. Then, given paths $\sigma_0$ and $\sigma_1$ from $x_0$ to $x_1$, $\sigma_0 * \overline{\sigma}_1$ is a loop based at $x_0$. Hence $\sigma_0 * \overline{\sigma}_1 \sim \varepsilon_{x_0}$ or, equivalently, $[\sigma_0][\overline{\sigma}_1] = [\varepsilon_{x_0}]$. Hence, $[\sigma_0][\overline{\sigma}_1][\sigma_1] = [\varepsilon_{x_0}][\sigma_1]$. But $[\sigma_0][\overline{\sigma}_1][\sigma_1] = [\sigma_0][\varepsilon_{x_1}] = [\sigma_0]$ and $[\varepsilon_{x_0}][\sigma_1] = [\sigma_1]$ and so $[\sigma_0] = [\sigma_1]$, i.e. $\sigma_0 \sim \sigma_1$.

‘$\Leftarrow$’ Suppose that that all paths from $x_0$ to $x_1$ are equivalent. Let $\tau$ be a path from $x_0$ to $x_1$. Then given a loop $\sigma$ at $x_0$, $\sigma * \tau$ is a path from $x_0$ to $x_1$ and so $\sigma * \tau \sim \tau$ or, equivalently, $[\sigma][\tau] = [\tau]$. Hence, $[\sigma][\tau][\overline{\tau}] = [\tau][\overline{\tau}]$. But $[\sigma][\overline{\tau}] = [\sigma][\varepsilon_{x_1}] = [\sigma]$ and $[\tau][\overline{\tau}] \sim [\varepsilon_{x_0}]$ so that $[\sigma] = [\varepsilon_{x_0}]$. Hence $\pi_1(X, x_0) = \{[\varepsilon_{x_0}]\}$, the trivial group, and so $X$ is simply connected.

4. ‘$\Rightarrow$’ Suppose that $\pi_1(X)$ is abelian. Suppose that $\rho_0$ and $\rho_1$ are paths from $x_0$ to $x_1$ and $\sigma$ is a loop at $x_0$. Then $u_{\rho_0}([\sigma]) = u_{\rho_1}([\sigma]) \iff [\overline{\rho}_0][\sigma][\rho_0] = [\overline{\rho}_1][\sigma][\rho_1] \iff [\overline{\sigma}]^{[\rho_0]}[\overline{\sigma}]^{[\rho_1]} = [\rho_0][\overline{\rho}_1][\sigma]$ (using $[\overline{\rho}_1] = [\rho_1]^{-1}$). But this final equality is true since $[\rho_0][\overline{\rho}_1]$ and $[\sigma] \in \pi_1(X, x_0)$ which is abelian. Hence $u_{\rho_0}([\sigma]) = u_{\rho_1}([\sigma])$ for all $[\sigma] \in \pi(X, x_0)$ and so $u_{\rho_0} = u_{\rho_1}$.

‘$\Leftarrow$’ Suppose that all paths from $x_0$ to $x_1$ induce the same isomorphism and that $\sigma$ and $\tau$ are loops based at $x_0$. [It may not be very clear how to
proceed but to use the data we have to find two paths from \( x_0 \) to \( x_1 \). One exists since \( X \) is path-connected and so this suggests using one of the loops to obtain a second path and then applying the resulting isomorphisms to the other loop. This works.] Let \( \rho \) be a path in \( X \) from \( x_0 \) to \( x_1 \). Then \( \tau \ast \rho \) is a second path from \( x_0 \) to \( x_1 \) so that \( u_{\tau \ast \rho}([\sigma]) = u_{\rho}([\sigma]) \). This gives us \((\tau)(\rho)^{-1}[\sigma](\tau)(\rho) = [\rho]^{-1}[\sigma][\rho]\) which gives \([\rho]^{-1}[\tau]^{-1}[\sigma][\tau][\rho] = [\rho]^{-1}[\sigma][\rho]\) which gives \([\sigma][\tau] = [\tau][\sigma]\) proving that \( \pi_1(X) \) is abelian.

5. (a) This is true. Given \([y] \in \pi_0(Y)\) then \( y = f(x) \) for some \( x \in X \) and \([y] = f_*([x])\).
   (b) This is false. For example the inclusion map \( i: \{0, 1\} \to [0, 1] \) is an injection but the first space has two path-components whereas the second has only one.
   (c) This is false. For example the function \( f: [0, 1) \cup \{2\} \to [0, 1] \) given by \( f(x) = x \) for \( x \in [0, 1) \) and \( f(2) = 1 \) is a continuous bijection but the first space has two path components whereas the second has only one.