A1.

(a) Define what is meant by a topology on a set $X$.

(b) Define what is meant by saying that a function $f: X \to Y$ between topological spaces is continuous. Define what is meant by saying that $f$ is a homeomorphism.

(c) Prove that the closed disc $D^2 = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$ with the usual topology is homeomorphic to the hemisphere $\{x = (x_1, x_2, x_3) \in S^2 \mid x_3 \geq 0\}$.

[Here $S^2$ denotes the unit sphere $\{x \in \mathbb{R}^3 \mid |x| = 1\}$ with the usual topology.]

[10 marks]

Solution

(a) Given a set $X$, a topology on $X$ is a collection $\tau$ of subsets of $X$ with the following properties:

(i) $\emptyset \in \tau$, $X \in \tau$;

(ii) the intersection of any two subsets in $\tau$ is in $\tau$:

$$U_1, U_2 \in \tau \Rightarrow U_1 \cap U_2 \in \tau;$$

(iii) the union of any collection of subsets in $\tau$ is in $\tau$:

$$U_\lambda \in \tau \text{ for all } \lambda \in \Lambda \Rightarrow \bigcup_{\lambda \in \Lambda} U_\lambda \in \tau.$$  

[5 marks, bookwork]

(b) $f: X \to Y$ is continuous if

$$V \text{ is open in } Y \Rightarrow f^{-1}(V) \text{ is open in } X$$

[1 marks, bookwork]

A homeomorphism is a continuous bijection with continuous inverse.

[2 marks, bookwork]

(c) A homeomorphism $f: \{x \in S^2 \mid x_3 \geq 0\} \to D^2$ is given by $f(x_1, x_2, x_3) = (x_1, x_2)$ with inverse $f^{-1}(y_1, y_2) = (y_1, y_2, \sqrt{1 - y_1^2 - y_2^2})$.

[2 marks, question set]

[Total: 10 marks]

The question was generally well done. Some people didn’t come up with the homeomorphisms in (c). In (b) sometimes only the (old) definition for the case of subspaces of $\mathbb{R}^n$ was given, but the question explicitly asks for the case of (general) topological spaces (i.e. there is no notion of distance or $\epsilon$-balls).
A2.

(a) Define what is meant by saying that a topological space $X$ is *path-connected*.

(b) What is meant by saying the path-connectedness is a *topological property*?

(c) Prove that path-connectedness is a topological property.

(d) Prove that $\{(x_1, x_2) \in \mathbb{R}^2 \mid |(x_1, x_2 - 1)| \leq 1 \text{ or } |(x_1, x_2 + 1)| \leq 1\} \subset \mathbb{R}^2$ (with the usual topology) is path-connected.

[10 marks]

**Solution**

(a) A *path* from $x_0$ to $x_1$ in $X$ is a continuous function $\sigma: [0, 1] \rightarrow X$ with $\sigma(0) = x_0$ and $\sigma(1) = x_1$. $X$ is said to be *path-connected* if, for each pair of points $x_0, x_1 \in X$, there is a path in $X$ from $x_0$ to $x_1$.

[3 marks, bookwork]

(b) Saying that path-connectedness is a *topological property* means that, if $X \cong Y$ are homeomorphic topological spaces, then $X$ is path connected if and only if $Y$ is path-connected.

[1 marks, bookwork]

(c) To prove this, suppose that $X$ is path-connected. Then, given two points $y_0, y_1 \in Y$ let $x_0, x_1 \in X$ be points such that $f(x_i) = y_i$ (these points exist since $f$ is a bijection). Since $X$ is path-connected there is a path $\sigma: I \rightarrow X$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. Then $f \circ \sigma: I \rightarrow Y$ is a path in $Y$ from $y_0$ to $y_1$ (since the composition of continuous maps is continuous). Hence, $Y$ is path-connected. Conversely, if $Y$ is path-connected then so is $X$ by the same argument (interchanging the roles of $X$ and $Y$).

[3 marks, bookwork]

(d) First observe that $0 \in X$. Now, there is a path from every point of $X$ to 0, which implies path-connectedness by composition of paths. Indeed, for $x = (x_1, x_2) \in X$ consider the path $\sigma(t) = tx$. Assume first, that $|(x_1, x_2 - 1)| \leq 1$ then one has for $|\sigma(t) - (0, 1)|$

$$|tx - (0, 1)| = |tx - t(0, 1) - (1 - t)(0, 1)| \leq |tx - t(0, 1)| + |(1 - t)(0, 1)|$$

$$= t \cdot |x - (0, 1)| + (1 - t)$$

$$\leq 1.$$ 

Hence $\sigma(t) \in X$ for $t \in [0, 1]$. Similarly for $|(x_1, x_2 + 1)| \leq 1$ one obtains $|\sigma(t) + (0, 1)| \leq 1$ and, hence, $\sigma(t) \in X$.

Hence, for to arbitrary points $x, y$ a connecting path is given by

$$\tau(s) = \begin{cases} 
(1 - 2s)x & s \in [0, 1/2] \\
(2s - 1)y & s \in [1/2, 1]. 
\end{cases}$$

[3 marks, new]
Except from part (d) the question was generally done well. Sometimes in (c) people showed that for two points in $Y$ of the form $f(x_0)$ and $f(x_1)$ there is a path. But you also need to refer to surjectivity of $f$ to see that all elements in $Y$ are of this form. For (d) some people constructed a path along the straight line between two points. This doesn’t work here as the subset is not convex and, hence, doesn’t contain the straight line between arbitrary points. A good point to start with was to sketch the subset in order to come up with a good guess for constructing a connecting path.

A3.

(a) Define what is meant by saying that a topological space is Hausdorff.

(b) Determine whether the set $S = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a, c\}, \{b\}, \{a, b, c\}\}$ is Hausdorff.

(c) Suppose that $X$ and $Y$ are topological spaces. Define the product topology on the Cartesian product $X \times Y$. [It is not necessary to prove that this is a topology.]

(d) Prove that if $\Delta \subset X \times X$ is closed in the product topology, then $X$ is Hausdorff.

[10 marks]

Solution

(a) The topological space $X$ is Hausdorff if, for each distinct pair of points $x, y \in X$, there exist open sets $U$ and $V$ in $X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. [2 marks, bookwork]

(b) This space is not Hausdorff because every open subset containing $a$ also contains $c$ and so open subsets as required cannot be found for $x = a$ and $y = c$. [3 marks, bookwork]

(c) The product topology on $X \times Y$ has a basis

\[ \{U \times V \mid U \text{ open in } X, V \text{ open in } Y\}, \]

i.e. the open sets consist of all unions of such sets. [3 marks, bookwork]

(d) Assume $\Delta$ is closed. Hence $X \times X \setminus \Delta$ is open. By definition of the product topology this means it is a union of open rectangles, i.e. sets of the form $U \times V \subset X \times X \setminus \Delta$ with $U$ and $V$ both open in $X$. Consider $x, y \in X$ with $x \neq y$ then $(x, y)$ lies outside the diagonal. Hence, is has to be contained in such a set

\[ U \times V \subset X \times X \setminus \Delta. \]

On the one hand this implies that $x \in U$ and $y \in V$. On the other hand $U \cap V = \emptyset$, since for $x \in U \cap V$ one would have $\Delta \ni (x, x) \in U \times V$. [2 marks, question set]
Problems occurred in (c), where people forgot that open subsets not only arise as open rectangles, but also as unions of such rectangles. For (d) quite a few people didn’t know that $\Delta$ was supposed to denote the diagonal, others did (indeed, the question was given in week 5 as an exercise with the solution being discussed in the tutorial). Nevertheless, in hindsight it would have been better to explicitly state the definition of $\Delta$ in the question. However, this issue was taken into account when marking the papers.

A4.

(a) Suppose that $X_1$ is a subspace of a topological space $X$. Define what is meant by saying that $X_1$ is a retract of $X$.

(b) Use the functorial properties of the fundamental group to prove that, if $X_1$ is a retract of $X$, then, for any $x_0 \in X_1$, the homomorphism induced by the inclusion map $i_* : \pi_1(X_1, x_0) \to \pi_1(X, x_0)$ is injective.

(c) Hence prove that $S^1$ is not a retract of the closed disc $D^2$.

[You may quote any fundamental groups that you need, without proof.]

Solution

(a) $X_1 \subset X$ is a retract of $X$ when there is a continuous map $r : X \to X_1$, such that $r(x) = x$ for $x \in X_1$. [3 marks, bookwork]

(b) By the functorial properties we have

$$r_* \circ i_* = (r \circ i)_* = (\text{id}_{X_1})_* = \text{id}_{\pi_1(X_1, x_0)} : \pi_1(X_1, x_0) \to \pi_1(X_1, x_0).$$

Since the composition of $r_*$ and $i_*$ is bijective $r_*$ must be surjective and $i_*$ must be injective. [4 marks, bookwork]

(c) We have $\pi_1(S^1, x_0) = \mathbb{Z}$ and $\pi_1(D^2, x_0) = 1$, the trivial group. But there is not injective map $\mathbb{Z} \to \{1\}$. Hence, $S^1$ cannot be a retract of $D^2$. [3 marks, bookwork]

[Total: 10 marks]

In (a) sometimes it was stated that $i \circ r$ has to be the identity. Which clearly cannot be the case except if $X = X_1$. [4 of 9] P.T.O.
B5.

(a) Suppose that $q : X \to Y$ is a surjection from a topological space $X$ to a set $Y$. Define the quotient topology on $Y$ determined by $q$. State the universal property of the quotient topology.

(b) Suppose that $f : X \to Z$ is a continuous surjection from a compact topological space $X$ to a Hausdorff topological space $Z$. Define an equivalence relation $\sim$ on $X$ so that $f$ induces a bijection $F : X/\sim \to Z$ from the identification space $X/\sim$ of this equivalence relation to $Z$. Prove that $F$ is a homeomorphism. [State clearly any general results which you use.]

(c) Prove that the quotient space $[0, 1] \times [0, 1]/\sim$ with $(0, s) \sim (1, s)$ is homeomorphic to the cylinder $[0, 1] \times S^1 \subset \mathbb{R}^3$. 

[15 marks]

Solution

(a) Given a topological space $(X, \tau)$ and a surjection $q : X \to Y$ the quotient topology on $Y$ is given by

$$\{ V \subset Y \mid q^{-1}(V) \in \tau \}.$$  

The universal property of the quotient topology is: $f : Y \to Z$ to a topological space $Z$ is continuous if and only if the composition $f \circ q : X \to Z$ is continuous.

[4 marks, bookwork]

(b) Given a continuous surjection $f : X \to Z$, define an equivalence relation on $X$ by $x \sim x' \iff f(x) = f(x')$. Then we may define $F : X/\sim \to Z$ by $F([x]) = f(x)$. Since $[x] = [x'] \iff x \sim x' \iff f(x) = f(x')$ (by the definition of the equivalence relation), the function $F$ is well-defined. Since $F([x]) = F([x']) \iff f(x) = f(x')$ (by the definition of the equivalence relation) it follows that $[x] = [x']$ and $F$ is injective. Since $f$ is a surjection, $y = f(x)$ for some $x \in X$ and so $y = F([x])$. Hence $F$ is a surjection. This shows that $F : X/\sim \to Z$ is a bijection. The map $F : X/\sim \to Z$ is continuous by the universal property since $F \circ q = f$ which is given as continuous, where $q : X \to X/\sim$ is the quotient map given by $q(x) = [x]$.

The space $X/\sim$ is compact since it is the continuous image of a compact set. Hence $F$ is a homeomorphism since it is a continuous bijection from a compact space to a Hausdorff space.

[7 marks, bookwork]

(c) To see this, define a surjection $f : I^2 \to I \times S^1$ by $f(x, y) = (y, \exp(2\pi ix))$ where we think of $S^1$ as $\{ z \in \mathbb{C} \mid |z| = 1 \}$ using the standard identification $\mathbb{C} \cong \mathbb{R}^2$. This function is continuous by the universal property of the product topology since the component functions are continuous. Now, $I \times S^1$ is Hausdorff (a subset of Euclidean space) and $I \times I$ is compact (a closed and bounded subset of Euclidean space). Now the result follows from (b).

[4 marks, bookwork]
B6.

(a) Define what is meant by a compact subset of a topological space and by a compact topological space.

(b) Prove that, if $f : X \to Y$ is a continuous function of topological spaces and $K \subset X$ is a compact subset, then $f(K)$ is a compact subset of $Y$.

(c) Given a non-compact Hausdorff space $(X, \tau)$ consider the set $X^* = X \cup \{\infty\}$ and the topology \[ \tau^* = \tau \cup \{(X \setminus C) \cup \{\infty\} \mid C \subset X \text{ compact}\} \]

Show that $(X^*, \tau^*)$ is compact.

[It is not necessary to prove that $\tau^*$ is a topology.]

Solution

(a) $K \subset X$ is compact if each cover of $K$ by open subsets of $X$ has a finite subcover.

If $X$ itself is a compact subset then $X$ is a compact space.

[3 marks, bookwork]

(b) Suppose that $\mathcal{F}$ is an open cover for $f(K)$. Let $f^{-1}(\mathcal{F}) = \{f^{-1}(V) \mid V \in \mathcal{F}\}$. Then $f^{-1}(\mathcal{F})$ is an over cover for $K$ since, given $a \in K$, $f(a) \in f(K)$ so that $f(a) \in V$ for some $V \in \mathcal{F}$. Hence $a \in f^{-1}(V)$ for some $V \in \mathcal{F}$.

Now, since $K$ is compact, $f^{-1}(\mathcal{F})$ has a finite subcover for $K$, $\{f^{-1}(V_1), f^{-1}(V_2), \ldots, f^{-1}(V_n)\}$. Thus, given $b \in f(K)$, $b = f(a)$ for some $a \in K$. Then $a \in f^{-1}(V_i)$ for some $i$, $1 \leq i \leq n$, so that $b = f(a) \in V_i$. Hence $\{V_1, V_2, \ldots, V_n\}$ is a finite subcover of $\mathcal{F}$ for $f(K)$.

Hence $f(K)$ is compact.

[6 marks, bookwork]

(c) Consider an open cover $\mathcal{F}$ of $X^*$. In order to contain $\infty$ it has to include at least one open subset $U_\infty$ of the form $X \setminus C \cup \{\infty\}$ where $C \subset X$ is compact. Now, $\mathcal{F}' = \{U \cap X \mid U \in \mathcal{F}\}$ is an open cover of $X$ (since $U$ and $X$ are open in $X^*$) and hence of $C$.

By compactness of $C$ a finite subcover $\{U_1 \cap X, \ldots, U_m \cap X\} \subset \mathcal{F}'$ suffices to cover $C$. But then one has the finite subcover $\{U_\infty, U_1, \ldots, U_m\} \subset \mathcal{F}$.

[6 marks, exercise set]
**B7.**

(a) Prove that, if the product \( \sigma_0 \ast \tau_0 \) of two paths \( \sigma_0 \) and \( \tau_0 \) in a topological space \( X \) is defined and the paths \( \sigma_1 \) and \( \tau_1 \) are homotopic to \( \sigma_0 \) and \( \tau_0 \) respectively, then the product \( \sigma_1 \ast \tau_1 \) is defined and is homotopic to \( \sigma_0 \ast \tau_0 \).

(b) Explain how a continuous function \( f: X \to Y \) induces a homomorphism \( f_*: \pi_1(X, x_0) \to \pi_1(X, f(x_0)) \). You should indicate why \( f_* \) is well-defined and why it is a homomorphism.

(c) Prove that, for topological spaces \( X \) and \( Y \) with points \( x_0 \in X, y_0 \in Y \), there is an isomorphism of groups

\[
\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).
\]

**Solution**

(a) Given homotopic paths \( H: \sigma_0 \sim \sigma_1 \) and \( K: \tau_0 \sim \tau_1 \) such that \( \sigma_0 \ast \tau_0 \) is defined. Then \( 1(1) = \sigma_0(1) = \tau_0(0) = \pi_1(0) \) and so the product \( \sigma_1 \ast \tau_1 \) is defined.

Suppose that \( H: \sigma_0 \sim \sigma_1 \) and \( K: \tau_0 \sim \tau_1 \). Then we may define a homotopy \( L: \sigma_0 \ast \tau_0 \sim \sigma_1 \ast \tau_1 \) by

\[
L(s, t) = \begin{cases} 
H(2s, t) & \text{for } 0 \leq s \leq 1/2 \text{ and } t \in I, \\
K(2s - 1, t) & \text{for } 1/2 \leq s \leq 1 \text{ and } t \in I.
\end{cases}
\]

This is well defined since, for \( s = 1/2, H(1, t) = x_1 = K(0, t) \). In addition, \( L \) is continuous by the Gluing Lemma since \([0, 1/2] \times I \) and \([1/2, 1] \times I \) are closed subsets of \( I^2 \)

[5 marks, bookwork]

(b) The function \( f_* \) is defined by \( f_*([\sigma]) = [f \circ \sigma] \). It is well-defined since, if \([\sigma_0] = [\sigma_1]\) then \( \sigma_0 \sim \sigma_1 \) and so there exists a homotopy \( H: \sigma_0 \sim \sigma_1 \). Then \( f \circ H: I^2 \to Y \) gives a homotopy \( f \circ \sigma_0 \sim f \circ \sigma_1 \) and so \([f \circ \sigma_0] = [f \circ \sigma_1]\).

To see that \( f_* \) is a homomorphism suppose that \([\sigma], [\tau] \in \pi_1(X, x_0)\). Then

\[
f_*([\sigma][\tau]) = f_*([\sigma \ast \tau]) = [f \circ (\sigma \ast \tau)]
\]

and

\[
f_*([\sigma])f_*([\tau]) = [f \circ \sigma][f \circ \tau] = [(f \circ \sigma) \ast (f \circ \tau)]
\]
and by writing out the formulae we see that \( f \circ (\sigma \ast \tau) = (f \circ \sigma) \ast (f \circ \tau) : I \rightarrow Y \). Hence, \( f_*([\sigma][\tau]) = f_*([\sigma])f_*([\tau]) \).

[5 marks, bookwork]

(c) Let \( p_1 : X \times Y \rightarrow X \) and \( p_2 : X \times Y \rightarrow Y \) be the projection maps. The function

\[
\pi_1(X \times Y, (x_0, y_0)) \rightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)
\]

given by \( \alpha \mapsto ((p_1)_*(\alpha), (p_2)_*(\alpha)) \) is an isomorphism. To see this we write down the inverse. Given a loop \( \sigma_1 \) in \( X \) based at \( x_0 \) and a loop \( \sigma_2 \) in \( Y \) based at \( y_0 \) then we may define a loop \( \sigma \) in \( X \times Y \) based at \( (x_0, y_0) \) by \( \sigma(s) = (\sigma_1(s), \sigma_2(s)) \). Then \( ([\sigma_1], [\sigma_2]) \mapsto [\sigma] \) is well-defined and provides the necessary inverse. Hence the given function is an isomorphism of groups.

[5 marks, question set]

[Total: 15 marks]

In part (a) sometimes the reference to the continuity of \( L \) via Gluing Lemma was missing. There was a typo in part (b) it should read \( f_* : \pi_0(X, x_0) \rightarrow \pi_0(Y, f(x_0)) \). This has been corrected during the exam. However, this shouldn’t have prevented anybody from solving this part and most of you probably didn’t even notice it. Only a few people succeeded with part (c).

B8.

(a) Define what is meant by the path-components of a topological space. [You may assume the definition of a path and properties of paths.]

(b) Prove that a continuous map of topological spaces \( f : X \rightarrow Y \) induces a map \( f_* : \pi_0(X) \rightarrow \pi_0(Y) \) between the sets of path-components, taking care to prove that your function is well-defined. Prove that if \( f \) is a homeomorphism then \( f_* \) is a bijection.

(c) A pair of distinct points \( \{p, q\} \) in a path-connected topological space \( X \) is called a cut-pair of type \( n \) when the subspace \( X \setminus \{p, q\} \) has \( n \) path-components. Prove that a homeomorphism \( f : X \rightarrow Y \) induces a bijection between the subsets of cut-pairs of type \( n \) for every \( n \in \mathbb{N} \).

(d) Hence show, using cut-pairs of type 3 or otherwise, that no two of the following subspaces of \( \mathbb{R}^2 \) with the usual topology are homeomorphic.

\[
\begin{align*}
(a) & \quad (b) & \quad (c)
\end{align*}
\]

[15 marks]
Solution

(a) Define an equivalence relation on \( X \) by \( x \sim x' \) if and only if there is a path in \( X \) from \( x \) to \( x' \). Then the path-components of \( X \) are the equivalence classes.

[2 marks, bookwork]

(b) Suppose that \( f : X \to Y \) is a continuous map. Then this induces a function \( f : \pi_0(X) \to \pi_0(Y) \) by \( f([x]) = [f(x)] \). This is well-defined because \( [x] = [x'] \) implies that \( x \sim x' \) so that there is a path \( \sigma : [0,1] \to X \) in \( X \) from \( x \) to \( x' \). Then \( f \circ \sigma : [0,1] \to Y \) is a path in \( Y \) from \( f(x) \) to \( f(x') \) and so \( [f(x)] = [f(x')] \).

[3 marks, bookwork]

If \( f \) is a homeomorphism then \( f_* \) is a bijection since the inverse \( g = f^{-1} : Y \to X \) induces a function \( g_* : \pi_0(Y) \to \pi_0(X) \) inverse to \( f_* \) since \( g_*(f_*([x])) = [g(f(x))] = [x] \) and \( f_*(g_*([y])) = [y] \).

[2 marks, bookwork]

(c) Suppose that \( f : X \to Y \) is a homeomorphism and \( \{p,q\} \) is a pair of distinct points in \( X \). Then \( f \) induces a homeomorphism \( X \setminus \{p,q\} \to Y \setminus \{f(p),f(q)\} \) and this induces a bijection \( f_* : \pi_0(X \setminus \{p,q\}) \to \pi_0(Y \setminus \{f(p),f(q)\}) \). Hence \( \{p,q\} \) is a cut-pair of type \( n \) in \( X \) if and only if \( \{f(p),f(q)\} \) is a cut-pair of type \( n \) in \( Y \).

[3 marks, exercise set]

(d) In space (i) there are two cut-pairs of type 3 (the intersection points of the line segments and the inner or out circle respectively). In space (ii) there is a unique cut-pair of type 3 (the two points at the ends of the diameter). In space (iii) there are infinitely many cut-pairs of type 3 (picking two arbitrary points on the radial line segments).

[5 marks, new]

[Total: 15 marks]

Part (a) and (b) were generally well done. Sometimes in the definition of \( f_* \) people mixed things up with the induced map on \( \pi_1 \) (instead of \( \pi_0 \)) which is also denoted by \( f_* \). In (c) people sometimes argued that there is a bijection between path-components of \( X \) and \( Y \), but one needs this statement for \( X \setminus \{p,q\} \) and \( X \setminus \{f(p),f(q)\} \). For this it is not enough that the restriction of \( f \) is a (continuous) bijection but one really needs that it is a homeomorphism. For part (d) almost everyone used the correct approach but sometimes people didn’t correctly identify cut pairs.