1 Topological Equivalence and Path-Connectedness

1.1 Definition. Suppose that $X$ and $Y$ are subsets of Euclidean spaces. A function $f: X \to Y$ is a topological equivalence or a homeomorphism if it is a continuous bijection such that the inverse $f^{-1}: Y \to X$ is also continuous. If such a homeomorphism exists then $X$ and $Y$ are topologically equivalent or homeomorphic, written $X \cong Y$.

1.2 Example. (a) The real line $\mathbb{R}$ and the open half line $(0, \infty) = \{ x \in \mathbb{R} \mid x > 0 \}$ are homeomorphic. A homeomorphism is given by $\exp: \mathbb{R} \to (0, \infty)$ with inverse $\log: (0, \infty) \to \mathbb{R}$.

(b) $X = \mathbb{R}^2 \setminus \{0\}$, the punctured plane, and $Y = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1 \}$, the infinite cylinder, are homeomorphic. A homeomorphism $f: X \to Y$ is given by

$$f(x_1, x_2) = \left(\frac{x_1}{|x|}, \frac{x_2}{|x|}, \log_e(|x|)\right)$$

with inverse $g: Y \to X$ given by

$$g(y_1, y_2, y_3) = e^{y_3}(y_1, y_2).$$

1.3 Exercise. (a) The punctured plane, $X = \mathbb{R}^2 \setminus \{0\}$, is homeomorphic to the complement of the unit disc, $Z = \{ x \in \mathbb{R}^2 \mid |x| > 1 \} = \mathbb{R}^2 \setminus D^2$ where $D^2 = \{ x \in \mathbb{R}^2 \mid |x| \leq 1 \}$.

(b) $S^1 = \{ x \in \mathbb{R}^2 \mid |x| = 1 \}$, the unit circle, and $T = \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| + |x_2| = 1 \}$, the diagonal square, are homeomorphic.

1.4 Problem. We prove that two subsets are homeomorphic by writing down a homeomorphism. How can we prove that two subsets are not homeomorphic?

1.5 Definition. A property $P$ of subsets of Euclidean spaces is a topological property when, if $X$ and $Y$ are homeomorphic subsets, then $X$ has property $P$ if and only if $Y$ has property $P$.

Thus, if $X$ has property $P$ and $Y$ does not have property $P$ then $X$ and $Y$ are not homeomorphic.
Path-connected subsets of Euclidean space

1.6 Definition. (a) Let \( X \) be a subset of some Euclidean space. A path in \( X \) is a continuous function \( \sigma: [0, 1] \rightarrow X \) where \( [0, 1] = \{ t \in \mathbb{R} | 0 \leq t \leq 1 \} \). The point \( \sigma(0) \) is the beginning point of the path and the point \( \sigma(1) \) is the terminal point of the path. We say that \( \sigma \) is a path in \( X \) from \( \sigma(0) \) to \( \sigma(1) \).

(b) The subset \( X \) is said to be path-connected if, for each pair of points \( x, x' \in X \), there is a path in \( X \) from \( x \) to \( x' \).

1.7 Proposition. The closed unit ball (or disc) \( D^n = \{ x \in \mathbb{R}^n | |x| \leq 1 \} \) in \( \mathbb{R}^n \) is path-connected.

Proof. Given \( x, x' \in D^n \) define \( \sigma: [0, 1] \rightarrow \mathbb{R}^n \) by
\[
\sigma(s) = x + s(x' - x) = (1 - s)x + sx'
\]
for \( s \in [0, 1] \). Then \( \sigma \) is continuous, \( \sigma(0) = x \) and \( \sigma(1) = x' \) so \( \sigma \) is a path in \( \mathbb{R}^n \) from \( x \) to \( x' \).

However, for \( 0 \leq s \leq 1 \), \( |\sigma(s)| = |(1 - s)x + sx'| \leq |(1 - s)x| + |sx'| \) (by the triangle inequality) = \( (1 - s)|x| + s|x'| \) (since \( s \geq 0 \) and \( 1 - s \geq 0 \)) \leq (1 - s) + s (since \( x, x' \in D^n \)) = 1, i.e. \( |\sigma(s)| \leq 1 \). Hence \( \sigma(s) \in D^n \) and so \( \sigma: [0, 1] \rightarrow D^n \) is a path in \( D^n \) from \( x \) to \( x' \).

Hence \( D^n \) is path-connected. \( \square \)

1.8 Exercise. The unit circle \( S^1 \) in \( \mathbb{R}^2 \) is path-connected.

1.9 Theorem. Let \( f: X \rightarrow Y \) be a continuous surjection where \( X \) and \( Y \) are subsets of Euclidean spaces. Then, if \( X \) is path-connected, so is \( Y \).

Proof. Exercise. \( \square \)

1.10 Corollary. Path-connectedness is a topological property.

Proof. Suppose that \( X \) and \( Y \) are homeomorphic subsets of Euclidean spaces. Then there is a homeomorphism \( f: X \rightarrow Y \). Then if \( X \) is path-connected so is \( Y \) by the Theorem since \( f \) is a continuous surjection. Conversely, if \( Y \) is path-connected then so is \( X \) since \( f^{-1}: Y \rightarrow X \) is a continuous surjection. Thus, \( X \) is path-connected if and only if \( Y \) is path-connected as required. \( \square \)

1.11 Proposition. The subset \( \mathbb{R} \setminus \{0\} \) is not path-connected and so \( \mathbb{R} \setminus \{0\} \not\sim S^1 \).
Proof. This is true because there is no path in $\mathbb{R} \setminus \{0\}$ from $-1$ to $1$. This may be proved by contradiction. Suppose, for contradiction, that $\sigma: [0, 1] \to \mathbb{R} \setminus \{0\}$ is a path from $-1$ to $1$ so that $\sigma(0) = -1$ and $\sigma(1) = 1$. Then $i \circ \sigma: [0, 1] \to \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a continuous function with values $-1$ and $1$ for which $0$ is not a value. This contradicts the intermediate value property of the function $\sigma$ (Theorem 0.23(b) in the Background Material) since $-1 < 0 < 1$ and so gives the necessary contradiction. Hence $\sigma$ cannot exist, as required and so $\mathbb{R} \setminus \{0\} \not\sim S^1$ since $S^1$ is path-connected and path-connectedness is a topological property.

1.12 Problem. Are $S^1$ and $[0, 1)$ homeomorphic? There is a continuous bijection $f: [0, 1) \to S^1$ defined by $f(x) = (\cos 2\pi x, \sin 2\pi x)$. More generally, is $S^1$ homeomorphic to any subset of $\mathbb{R}$?
Path-components

1.13 Definition. Suppose that $X$ is a subset of a Euclidean space.

(a) Given $x \in X$, we may define a path $\varepsilon_x : [0, 1] \to X$ by
$$\varepsilon_x(s) = x \text{ for } 0 \leq s \leq 1.$$ 
This is called the constant path at $x$.

(b) Given a path $\sigma : [0, 1] \to X$ in $X$ we may define a path
$$\bar{\sigma}(s) = \sigma(1 - s) \text{ for } 0 \leq s \leq 1.$$ 
This is called the reverse path of $\sigma$ and is a path from $\sigma(1)$ to $\sigma(0)$.

(c) Given paths $\sigma_1 : [0, 1] \to X$ and $\sigma_2 : [0, 1] \to X$ in $X$ such that $\sigma_1(1) = \sigma_2(0)$ we may define a path $\sigma_1 * \sigma_2 : [0, 1] \to X$ by
$$\sigma_1 * \sigma_2(s) = \begin{cases} \sigma_1(2s) & \text{for } 0 \leq s \leq 1/2, \\ \sigma_2(2s - 1) & \text{for } 1/2 \leq s \leq 1. \end{cases}$$ 
This is called the product of the paths $\sigma_1$ and $\sigma_2$ and is a path from $\sigma_1(0)$ to $\sigma_2(1)$.

[Note that $\sigma_1 * \sigma_2$ is well-defined and continuous at $t = 1/2$ by the conditions on $\sigma_1$ and $\sigma_2$.]

1.14 Proposition. Given $X$, a subset of a Euclidean space, we may define an equivalence relation on $X$ as follows: for $x, x' \in X$, $x \sim x'$ if and only if there is a path in $X$ from $x$ to $x'$.

Proof. We check the conditions for an equivalence relation (Definition 0.15).

The reflexive property. For each point $x \in X$, $x \sim x$ using the constant path $\varepsilon_x$.

The symmetric property. Suppose that $x$ and $x' \in X$ such that $x \sim x'$. Then there is a path $\sigma$ in $X$ from $x$ to $x'$. The reverse path $\bar{\sigma}$ is then a path in $X$ from $x'$ to $x$ and so $x' \sim x$ as required.

The transitive property. Suppose that $x, x'$ and $x'' \in X$ such that $x \sim x'$ and $x' \sim x''$. This means that there is a paths $\sigma_1$ in $X$ from $x$ to $x'$ and a path $\sigma_2$ in $X$ from $x'$ to $x''$. Then the product path $\sigma_1 * \sigma_2$ is a path in $X$ from $x$ to $x''$ and so $x \sim x''$ as required. \qed
1.15 Definition. Given $X$, a subset of a Euclidean space, the equivalence classes of the equivalence relation in Proposition 1.14 are called the path-components of $X$. We write $\pi_0(X)$ for the set of path-components of $X$ and $[x]$ for the path-component of a point $x \in X$.

1.16 Example. $\pi_0(\mathbb{R} \setminus \{0\}) = \{(-\infty, 0), (0, \infty)\}$.

1.17 Proposition. Homeomorphic sets have the same number of path-components.

Proof. Suppose that $X$ and $Y$ are homeomorphic subsets of Euclidean spaces. Then there is a homeomorphism $f : X \to Y$. It will be shown that this continuous function induces a bijection $f_* : \pi_0(X) \to \pi_0(Y)$ by $f_*([x]) = [f(x)]$. This implies that $\pi_0(X)$ and $\pi_0(Y)$ have the same cardinality which is what we have proved.

The function $f_*$ is well-defined because, if $[x] = [x']$ then $x \sim x'$ and so there is a path $\sigma : [0, 1] \to X$ in $X$ from $x$ to $x'$. It follows that $f \circ \sigma : [0, 1] \to Y$ is a path in $Y$ from $f(x)$ to $f(x')$ and so $f(x) \sim f(x')$, i.e. $[f(x)] = [f(x')]$.

The function $f_*$ is a bijection since it is easily checked that $(f^{-1})_* : \pi_0(Y) \to \pi_0(X)$, the function induced by the inverse $f^{-1} : Y \to X$, is an inverse for $f_*$ (Exercise). □

Cut-points in subsets of Euclidean space

1.18 Definition. Suppose that $X$ is a subset of some Euclidean space. Then a point $p \in X$ is called a cut-point of type $n$ of $X$ or an $n$-point of $X$ if its complement $X \setminus \{p\}$ has $n$ path-components.

1.19 Example. (a) In $[0, 1)$ each $x \in (0, 1)$ is a 2-point and 0 is a 1-point.

(b) In the subset of $\mathbb{R}^2$ given by the coordinate axes, $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0 \text{ or } x_2 = 0\}$, $(0, 0)$ is a 4-point whereas all other points are 2-points.

(c) In $S^1$ every point is a 1-point.

1.20 Theorem. Homeomorphic sets have the same number of cut-points of each type.

Proof. Let $X$ and $Y$ be homeomorphic subsets of Euclidean spaces. Then there is a homeomorphism $f : X \to Y$. Suppose that $p \in X$ is an $n$-point of $X$. Then $f$ induces a homeomorphism $X \setminus \{p\} \to Y \setminus \{f(p)\}$ and so these
subsets have the same number of path-components by Proposition 1.17. Hence \( f(p) \) is an \( n \)-point of \( Y \).

This shows that \( f \) induces a bijection between the \( n \)-points of \( X \) and the \( n \)-points of \( Y \) and so they must have the same number of \( n \)-points.  

**1.21 Example.** \([0,1)\) and \( S^1 \) are not homeomorphic since \([0,1)\) has some 2-points (all of its points apart from 0) whereas \( S^1 \) has none.

### Other applications of path-connectness

**1.22 Theorem (The Brouwer Fixed Point Theorem in dimension 1).** Suppose that \( f: [-1,1] \to [-1,1] \) is a continuous map. Then \( f \) has a fixed point, i.e. there exists a point \( t \in [-1,1] \) such that \( f(t) = t \).

**Proof.** Suppose for contradiction that \( f \) does not have a fixed point. Then \( f(t) \neq t \) for all \( t \in [-1,1] \). Thus we may define a function \( g: [-1,1] \to \{-1,1\} \) by \( g(t) = (f(t) - t)/|f(t) - t| \). This is a continuous function from basic real analysis. However, since \( f(-1) > -1 \) and \( f(1) < 1 \) it follows that \( g(-1) = 1 \) and \( g(1) = -1 \). Hence \( g \) is a surjection. Hence, by Proposition 1.9, \( \{-1,1\} \) path-connected which contradicts the Intermediate Value Theorem (as in the proof of Proposition 1.11). Hence \( f \) has a fixed point.  

**1.23 Theorem (The Borsuk-Ulam Theorem in dimension 1).** Suppose that \( f: S^1 \to \mathbb{R} \) is a continuous function. Then there is a point \( x \in S^1 \) such that \( f(x) = f(-x) \).

**Proof.** Exercise. Try a similar proof to that of Theorem 1.22.  

**1.24 Definition.** A subset \( A \subset \mathbb{R}^n \) is **bounded** if there is a real number \( R \) such that \( x \in A \implies |x| \leq R \).

**1.24 Theorem (The Pancake Theorem).** Let \( A \) and \( B \) be bounded subsets of \( \mathbb{R}^2 \). Then there is a (straight) line in \( \mathbb{R}^2 \) which divides each of \( A \) and \( B \) in half by area.

**Remark.** The statement of this result assumes that \( A \) and \( B \) each have a well-defined area. In this course we ignore the technical difficulties associated with defining the area of a subset of \( \mathbb{R}^2 \) (the subject of integration and measure theory).

**Outline Proof.** Since \( A \) and \( B \) are bounded there is a real number \( R \) such that \( a \in A \Rightarrow |a| \leq R \) and \( x \in B \Rightarrow |x| \leq R \).
Suppose that \( x \in S^1 \). For \( t \in [-R, R] \) let \( L_{xt} \) denote the straight line through \( tx \) perpendicular to \( x \). Let \( v(t) \in [0, 1] \) be the proportion of the area of \( A \) on the same side of \( L_{xt} \) as \( R \). Then \( v: [-R, R] \to [0, 1] \) is a continuous decreasing function with \( v(-R) = 1 \) and \( v(R) = 0 \). By the Intermediate Value Theorem there exists \( t \in [-R, R] \) such that \( v(t) = 1/2 \). This \( t \) may not be unique but it is not difficult to show that \( v^{-1}(1/2) = \{ t \mid v(t) = 1/2 \} = [\alpha, \beta] \), a closed interval. Let \( f_A(x) = (\alpha + \beta)/2 \). Then the line \( L_{x,f_A(x)} \) bisects \( A \).

The function \( f_A: S^1 \to \mathbb{R} \) can be shown to be continuous. Furthermore \( f_A(-x) = -f_A(x) \) (since \( L_{x,f_A(x)} \) and \( L_{x,f_A(-x)} \) are the same line so that \( f_A(x)x = f_A(-x)(-x) \)).

Similarly, using the region \( B \), we may define a continuous function \( f_B: S^1 \to \mathbb{R} \) such that \( f_B(-x) = -f_B(x) \) and \( L_{x,f_B(x)} \) bisects \( B \).

Let the continuous function \( f: S^1 \to \mathbb{R} \) be given by \( f(x) = f_A(x) - f_B(x) \).

By the Borsuk-Ulam Theorem, there exists \( x_0 \in S^1 \) such that \( f(x_0) = f(-x_0) \). But \( f(-x_0) = f_A(-x_0) - f_B(-x_0) = -f_A(x_0) + f_B(x_0) = -f(x_0) \). Hence \( f(x_0) = -f(x_0) \) so that \( f(x_0) = 0 \). This means that \( f_A(x_0) - f_B(x_0) = 0 \) so that \( f_A(x_0) = f_B(x_0) \).

From the definition of \( f_A \) and \( f_B \) it follows that the line \( L_{x_0,f_A(x_0)} = L_{x_0,f_B(x_0)} \) bisects both of \( A \) and \( B \) and so is the line whose existence is the claim of the theorem. \( \square \)