1 Topological Equivalence and Path-Connectedness

1.1 Definition. Suppose that $X$ and $Y$ are subsets of Euclidean spaces. A function $f: X \to Y$ is a topological equivalence or a homeomorphism if it is a continuous bijection such that the inverse $f^{-1}: Y \to X$ is also continuous. If such a homeomorphism exists then $X$ and $Y$ are topologically equivalent or homeomorphic, written $X \cong Y$.

1.2 Example. (a) The real line $\mathbb{R}$ and the open half line $(0, \infty) = \{ x \in \mathbb{R} \mid x > 0 \}$ are homeomorphic. A homeomorphism is given by $\exp: \mathbb{R} \to (0, \infty)$ with inverse $\log_e: (0, \infty) \to \mathbb{R}$.

(b) $X = \mathbb{R}^2 \setminus \{0\}$, the punctured plane, and $Y = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1 \}$, the infinite cylinder, are homeomorphic. A homeomorphism $f: X \to Y$ is given by

$$f(x_1, x_2) = \left( \frac{x_1}{|x|}, \frac{x_2}{|x|}, \log_e(|x|) \right)$$

with inverse $g: Y \to X$ given by

$$g(y_1, y_2, y_3) = e^{y_3}(y_1, y_2).$$

1.3 Exercise. (a) The punctured plane, $X = \mathbb{R}^2 \setminus \{0\}$, is homeomorphic to the complement of the unit disc, $Z = \{ x \in \mathbb{R}^2 \mid |x| > 1 \} = \mathbb{R}^2 \setminus D^2$ where $D^2 = \{ x \in \mathbb{R}^2 \mid |x| \leq 1 \}$.

(b) $S^1 = \{ x \in \mathbb{R}^2 \mid |x| = 1 \}$, the unit circle, and $T = \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| + |x_2| = 1 \}$, the diagonal square, are homeomorphic.

1.4 Problem. We prove that two subsets are homeomorphic by writing down a homeomorphism. How can we prove that two subsets are not homeomorphic?

1.5 Definition. A property $P$ of subsets of Euclidean spaces is a topological property when, if $X$ and $Y$ are homeomorphic subsets, then $X$ has property $P$ if and only if $Y$ has property $P$.

Thus, if $X$ has property $P$ and $Y$ does not have property $P$ then $X$ and $Y$ are not homeomorphic.
Path-connected subsets of Euclidean space

1.6 Definition. (a) Let $X$ be a subset of some Euclidean space. A path in $X$ is a continuous function $\sigma : [0, 1] \to X$ where $[0, 1] = \{ t \in \mathbb{R} \mid 0 \leq t \leq 1 \}$. The point $\sigma(0)$ is the beginning point of the path and the point $\sigma(1)$ is the terminal point of the path. We say that $\sigma$ is a path in $X$ from $\sigma(0)$ to $\sigma(1)$.

(b) The subset $X$ is said to be path-connected if, for each pair of points $x, x' \in X$, there is a path in $X$ from $x$ to $x'$.

1.7 Proposition. The closed unit ball (or disc) $D^n = \{ x \in \mathbb{R}^n \mid |x| \leq 1 \}$ in $\mathbb{R}^n$ is path-connected.

Proof. Given $x, x' \in D^n$ define $\sigma : [0, 1] \to \mathbb{R}^n$ by

$$\sigma(s) = x + s(x' - x) = (1 - s)x + sx'$$

for $s \in [0, 1]$. Then $\sigma$ is continuous, $\sigma(0) = x$ and $\sigma(1) = x'$ so $\sigma$ is a path in $\mathbb{R}^n$ from $x$ to $x'$.

However, for $0 \leq s \leq 1$, $|\sigma(s)| = |(1 - s)x + sx'| \leq |(1 - s)x| + |sx'|$ (by the triangle inequality) $= (1 - s)||x|| + s||x'||$ (since $s \geq 0$ and $1 - s \geq 0$) $\leq (1 - s) + s$ (since $x, x' \in D^n = 1$, i.e. $|\sigma(s)|| \leq 1$. Hence $\sigma(s) \in D^n$ and so $\sigma : [0, 1] \to D^n$ is a path in $D^n$ from $x$ to $x'$.

Hence $D^n$ is path-connected. \hfill \Box

1.8 Exercise. The unit circle $S^1$ in $\mathbb{R}^2$ is path-connected.

1.9 Theorem. Let $f : X \to Y$ be a continuous surjection where $X$ and $Y$ are subsets of Euclidean spaces. Then, if $X$ is path-connected, so is $Y$.

Proof. Exercise. \hfill \Box

1.10 Corollary. Path-connectedness is a topological property.

Proof. Suppose that $X$ and $Y$ are homeomorphic subsets of Euclidean spaces. Then there is a homeomorphism $f : X \to Y$. Then if $X$ is path-connected so is $Y$ by the Theorem since $f$ is a continuous surjection. Conversely, if $Y$ is path-connected then so is $X$ since $f^{-1} : Y \to X$ is a continuous surjection. Thus, $X$ is path-connected if and only if $Y$ is path-connected as required. \hfill \Box

1.11 Proposition. The subset $\mathbb{R} \setminus \{0\}$ is not path-connected and so $\mathbb{R} \setminus \{0\} \not\sim S^1$.  

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Proof. This is true because there is no path in \( \mathbb{R} \setminus \{0\} \) from \(-1\) to \(1\). This may be proved by contradiction. Suppose, for contradiction, that \( \sigma : [0, 1] \to \mathbb{R} \setminus \{0\} \) is a path from \(-1\) to \(1\) so that \( \sigma(0) = -1 \) and \( \sigma(1) = 1 \). Then \( i \circ \sigma : [0, 1] \to \mathbb{R} \setminus \{0\} \to \mathbb{R} \) is a continuous function with values \(-1\) and \(1\) for which \(0\) is not a value. This contradicts the intermediate value property of the function \( \sigma \) (Theorem 0.23(b) in the Background Material) since \(-1 < 0 < 1\) and so gives the necessary contradiction. Hence \( \sigma \) cannot exist, as required and so \( \mathbb{R} \setminus \{0\} \not\sim S^1 \) since \( S^1 \) is path-connected and path-connectedness is a topological property.

\( \square \)

1.12 Problem. Are \( S^1 \) and \([0, 1)\) homeomorphic? There is a continuous bijection \( f : [0, 1) \to S^1 \) defined by \( f(x) = (\cos 2\pi x, \sin 2\pi x) \). More generally, is \( S^1 \) homeomorphic to any subset of \( \mathbb{R} \)?
Path-components

1.13 Definition. Suppose that $X$ is a subset of a Euclidean space.

(a) Given $x \in X$, we may define a path $\varepsilon_x : [0, 1] \to X$ by

$$\varepsilon_x(s) = x \quad \text{for } 0 \leq s \leq 1.$$ 

This is called the constant path at $x$.

(b) Given a path $\sigma : [0, 1] \to X$ in $X$ we may define a path

$$\sigma(s) = \sigma(1 - s) \quad \text{for } 0 \leq s \leq 1.$$ 

This is called the reverse path of $\sigma$ and is a path from $\sigma(1)$ to $\sigma(0)$.

(c) Given paths $\sigma_1 : [0, 1] \to X$ and $\sigma_2 : [0, 1] \to X$ in $X$ such that $\sigma_1(1) = \sigma_2(0)$ we may define a path $\sigma_1 * \sigma_2 : [0, 1] \to X$ by

$$\sigma_1 * \sigma_2(s) = \begin{cases} \sigma_1(2s) & \text{for } 0 \leq s \leq 1/2, \\ \sigma_2(2s - 1) & \text{for } 1/2 \leq s \leq 1. \end{cases}$$ 

This is called the product of the paths $\sigma_1$ and $\sigma_2$ and is a path from $\sigma_1(0)$ to $\sigma_2(1)$.

[Note that $\sigma_1 * \sigma_2$ is well-defined and continuous at $t = 1/2$ by the conditions on $\sigma_1$ and $\sigma_2$.]

1.14 Proposition. Given $X$, a subset of a Euclidean space, we may define an equivalence relation on $X$ as follows: for $x, x' \in X$, $x \sim x'$ if and only if there is a path in $X$ from $x$ to $x'$.

Proof. We check the conditions for an equivalence relation (Definition 0.15).

The reflexive property. For each point $x \in X$, $x \sim x$ using the constant path $\varepsilon_x$.

The symmetric property. Suppose that $x$ and $x' \in X$ such that $x \sim x'$. Then there is a path $\sigma$ in $X$ from $x$ to $x'$. The reverse path $\bar{\sigma}$ is then a path in $X$ from $x'$ to $x$ and so $x' \sim x$ as required.

The transitive property. Suppose that $x, x'$ and $x'' \in X$ such that $x \sim x'$ and $x' \sim x''$. This means that there is a paths $\sigma_1$ in $X$ from $x$ to $x'$ and a path $\sigma_2$ in $X$ from $x'$ to $x''$. Then the product path $\sigma_1 * \sigma_2$ is a path in $X$ from $x$ to $x''$ and so $x \sim x''$ as required. 

\[ \square \]
1.15 Definition. Given $X$, a subset of a Euclidean space, the equivalence classes of the equivalence relation in Proposition 1.14 are called the path-components of $X$. We write $\pi_0(X)$ for the set of path-components of $X$ and $[x]$ for the path-component of a point $x \in X$.

1.16 Example. $\pi_0(\mathbb{R} \setminus \{0\}) = \{(-\infty, 0), (0, \infty)\}$.

1.17 Proposition. Homeomorphic sets have the same number of path-components.

Proof. Suppose that $X$ and $Y$ are homeomorphic subsets of Euclidean spaces. Then there is a homeomorphism $f : X \to Y$. It will be shown that this continuous function induces a bijection $f_* : \pi_0(X) \to \pi_0(Y)$ by $f_*([x]) = [f(x)]$. This implies that $\pi_0(X)$ and $\pi_0(Y)$ have the same cardinality which is what we have prove.

The function $f_*$ is well-defined because, if $[x] = [x']$ then $x \sim x'$ and so there is a path $\sigma : [0, 1] \to X$ in $X$ from $x$ to $x'$. It follows that $f \circ \sigma : [0, 1] \to Y$ is a path in $Y$ from $f(x)$ to $f(x')$ and so $f(x) \sim f(x')$, i.e. $[f(x)] = [f(x')]$. The function $f_*$ is a bijection since it is easily checked that $(f^{-1})_* : \pi_0(Y) \to \pi_0(X)$, the function induced by the inverse $f^{-1} : Y \to X$, is an inverse for $f_*$ (Exercise). □

Cut-points in subsets of Euclidean space

1.18 Definition. Suppose that $X$ is a subset of some Euclidean space. Then a point $p \in X$ is called a cut-point of type $n$ of $X$ or an $n$-point of $X$ if its complement $X \setminus \{p\}$ has $n$ path-components.

1.19 Example. (a) In $[0, 1)$ each $x \in (0, 1)$ is a 2-point and 0 is a 1-point.

(b) In the subset of $\mathbb{R}^2$ given by the coordinate axes, $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0 \text{ or } x_2 = 0\}$, $(0, 0)$ is a 4-point whereas all other points are 2-points.

(c) In $S^1$ every point is a 1-point.

1.20 Theorem. Homeomorphic sets have the same number of cut-points of each type.

Proof. Let $X$ and $Y$ be homeomorphic subsets of Euclidean spaces. Then there is a homeomorphism $f : X \to Y$. Suppose that $p \in X$ is an $n$-point of $X$. Then $f$ induces a homeomorphism $X \setminus \{p\} \to Y \setminus \{f(p)\}$ and so these
subsets have the same number of path-components by Proposition 1.17. Hence \( f(p) \) is an \( n \)-point of \( Y \).
This shows that \( f \) induces a bijection between the \( n \)-points of \( X \) and the \( n \)-points of \( Y \) and so they must have the same number of \( n \)-points. \( \square \)

1.21 Example. \([0, 1)\) and \( S^1 \) are not homeomorphic since \([0, 1)\) has some 2-points (all of its points apart from 0) whereas \( S^1 \) has none.

Other applications of path-connectness

1.22 Theorem (The Brouwer Fixed Point Theorem in dimension 1). Suppose that \( f: [-1, 1] \to [-1, 1] \) is a continuous map. Then \( f \) has a fixed point, i.e. there exists a point \( t \in [-1, 1] \) such that \( f(t) = t \).

Proof. Suppose for contradiction that \( f \) does not have a fixed point. Then \( f(t) \neq t \) for all \( t \in [-1, 1] \). Thus we may define a function \( g: [-1, 1] \to \{-1, 1\} \) by \( g(t) = (f(t) - t)/|f(t) - t| \). This is a continuous function from basic real analysis. However, since \( f(-1) > -1 \) and \( f(1) < 1 \) it follows that \( g(-1) = 1 \) and \( g(1) = -1 \). Hence \( g \) is a surjection. Hence, by Proposition 1.9, \( \{-1, 1\} \) path-connected which contradicts the Intermediate Value Theorem (as in the proof of Proposition 1.11). Hence \( f \) has a fixed point. \( \square \)

1.23 Theorem (The Borsuk-Ulam Theorem in dimension 1). Suppose that \( f: S^1 \to \mathbb{R} \) is a continuous function. Then there is a point \( x \in S^1 \) such that \( f(x) = f(-x) \).

Proof. Exercise. Try a similar proof to that of Theorem 1.22. \( \square \)

1.24 Definition. A subset \( A \subset \mathbb{R}^n \) is bounded if there is a real number \( R \) such that \( x \in A \implies |x| \leq R \).

1.24 Theorem (The Pancake Theorem). Let \( A \) and \( B \) be bounded subsets of \( \mathbb{R}^2 \). Then there is a (straight) line in \( \mathbb{R}^2 \) which divides each of \( A \) and \( B \) in half by area.

Remark. The statement of this result assumes that \( A \) and \( B \) each have a well-defined area. In this course we ignore the technical difficulties associated with defining the area of a subset of \( \mathbb{R}^2 \) (the subject of integration and measure theory).

Outline Proof. Since \( A \) and \( B \) are bounded there is a real number \( R \) such that \( a \in A \Rightarrow |a| \leq R \) and \( x \in B \Rightarrow |x| \leq R \).
Suppose that \( x \in S^1 \). For \( t \in [-R, R] \) let \( L_{x,t} \) denote the straight line through \( tx \) perpendicular to \( x \). Let \( v(t) \in [0, 1] \) be the proportion of the area of \( A \) on the same side of \( L_{x,t} \) as \( Rx \). Then \( v: [-R, R] \to [0, 1] \) is a continuous decreasing function with \( v(-R) = 1 \) and \( v(R) = 0 \). By the Intermediate Value Theorem there exists \( t \in [-R, R] \) such that \( v(t) = 1/2 \). This \( t \) may not be unique but it is not difficult to show that \( v^{-1}(1/2) = \{ t \mid v(t) = 1/2 \} = [\alpha, \beta] \), a closed interval. Let \( f_A(x) = (\alpha + \beta)/2 \). Then the line \( L_{x,f_A(x)} \) bisects \( A \).

The function \( f_A: S^1 \to \mathbb{R} \) can be shown to be continuous. Furthermore \( f_A(-x) = -f_A(x) \) (since \( L_{x,f_A(x)} \) and \( L_{x,f_A(-x)} \) are the same line so that \( f_A(x)x = f_A(-x)(-x) \)).

Similarly, using the region \( B \), we may define a continuous function \( f_B: S^1 \to \mathbb{R} \) such that \( f_B(-x) = -f_B(x) \) and \( L_{x,f_B(x)} \) bisects \( B \).

Let the continuous function \( f: S^1 \to \mathbb{R} \) be given by \( f(x) = f_A(x) - f_B(x) \). By the Borsuk-Ulam Theorem, there exists \( x_0 \in S^1 \) such that \( f(x_0) = f(-x_0) \). But \( f(-x_0) = f_A(-x_0) - f_B(-x_0) = -f_A(x_0) + f_B(x_0) = -f(x_0) \).

Hence \( f(x_0) = -f(x_0) \) so that \( f(x_0) = 0 \). This means that \( f_A(x_0) - f_B(x_0) = 0 \) so that \( f_A(x_0) = f_B(x_0) \).

From the definition of \( f_A \) and \( f_B \) it follows that the line \( L_{x_0,f_A(x_0)} = L_{x_0,f_B(x_0)} \) bisects both of \( A \) and \( B \) and so is the line whose existence is the claim of the theorem. \( \square \)