Problems 3: Subspaces and Product Spaces

1. Suppose that \( X_2 \subset X_1 \subset X \) where \( X \) is a topological space. Prove that the subspace topology on \( X_2 \) induced by the topology on \( X \) is the same as the subspace topology on \( X_2 \) induced by the subspace topology on \( X_1 \).

2. Suppose that \( X \subset \mathbb{R}^n \). Prove that the subspace topology on \( X \) induced by the usual topology on \( \mathbb{R}^n \) is the same as the usual topology on \( X \) (given by Definition 2.5). [Remark 3.4]

3. Prove Proposition 3.5: For a subspace \( X_1 \) of a topological space \( X \), the closed subsets of \( X_1 \) are all subsets of the form \( A \cap X_1 \) where \( A \) is a closed subset of \( X \).

4. Prove that all open subsets of a subspace \( X_1 \) of a topological space \( X \) are open in \( X \) if and only if \( X_1 \) is open in \( X \).

Deduce the following version of the Gluing Lemma (cf. Theorem 3.7). Suppose that \( X_1 \) and \( X_2 \) are open subspaces of \( X \) such that \( X = X_1 \cup X_2 \) and \( f_i : X_i \rightarrow Y \) are continuous functions to a topological space \( Y \) \((i = 1, 2)\) such that \( f_1(x) = f_2(x) \) for \( x \in X_1 \cap X_2 \). Then the function \( f : X \rightarrow Y \) defined by \( f(x) = f_i(x) \) for \( x \in X_i \) is well-defined and continuous.

5. Find subspaces \( X_1 \) and \( X_2 \) of a topological space \( X \) and continuous functions \( f_i : X_i \rightarrow Y \), to a topological space \( Y \) such that \( f_1(x) = f_2(x) \) for all \( x \in X_1 \cap X_2 \) such that the function \( f : X \rightarrow Y \) given by \( f(x) = f_i(x) \) for \( x \in X_i \) is not continuous. [Hint: What is the simplest non-continuous function you can think of? Find some subspaces on which its restriction is continuous.]

6. Prove that the product space \( \mathbb{R} \times S^1 \) (the infinite cylinder) is homeomorphic to the punctured plane \( \mathbb{R}^2 - \{0\} \).
7. Given subsets $Y_1 \subset X_1$ and $Y_2 \subset X_2$ of topological spaces $X_1$ and $X_2$ prove that the following two topologies on $Y_1 \times Y_2$ are the same:

(i) $Y_1 \times Y_2$ is a subspace of the product space $X_1 \times X_2$;
(ii) $Y_1 \times Y_2$ is a product of the subspaces $Y_1$ (of $X_1$) and $Y_2$ (of $X_2$).

[Remark 3.12(c)]

[Hint: Consider the identity function $(Y_1 \times Y_2, \tau_1) \to (Y_1 \times Y_2, \tau_2)$ where $\tau_1$ and $\tau_2$ are the two topologies defined in the question. Use the universal properties of the product topology and the subspace topology to show that this identity function and its inverse are both continuous.]

8. Prove that the product space $\mathbb{R} \times \{0, 1\}$, where $\mathbb{R}$ has the usual topology and $\{0, 1\}$ has the indiscrete topology, is path-connected.

9. Suppose that $X_1$ and $X_2$ are disjoint topological spaces (i.e. $X_1 \cap X_2 = \emptyset$). Prove that we may define a topology on the union $X_1 \cup X_2$ by: $U \subset X_1 \cup X_2$ is open if and only if $U \cap X_1$ is open in $X_1$ and $U \cap X_2$ is open in $X_2$. With this topology $X_1 \cup X_2$ is called the disjoint union of the topological spaces (or sometimes the coproduct), often denoted $X_1 \sqcup X_2$.

Prove the following universal property of the disjoint union topology. A function $f : X_1 \cup X_2 \to Y$ to a topological space $Y$ is continuous if and only if the restricted functions $f|X_1 = f \circ i_1 : X_1 \to Y$ and $f|X_2 = f \circ i_2 : X_2 \to Y$ are continuous (where $i_j : X_j \to X_1 \sqcup X_2$ for $j = 1, 2$ are the inclusion maps).