Three hours

THE UNIVERSITY OF MANCHESTER

ALGEBRAIC TOPOLOGY

28 May 2015

09:45 — 12:45

Answer **ALL** FOUR questions in Section A (40 marks in total). Answer **THREE** of the FOUR questions in Section B (45 marks in total). Answer **ALL** THREE questions in Section C (50 marks in total). If more than THREE questions from Section B are attempted then credit will be given for the best THREE answers.

Electronic calculators are permitted, provided they cannot store text.

SECTION A

Answer **ALL** FOUR questions.

A1. (a) Define what is meant by a *geometric simplicial surface.*

[The notions of *triangle* in $\mathbb{R}^n$, and its *vertices* and *edges* may be used without definition.]

(b) Define what is meant by the statement that a simplicial surface is *orientable.*

(c) Define what is meant by the statement that the orientability of a simplicial surface is a *topological invariant.*

[10 marks]

A2. (a) Define what it meant by a *geometric simplicial complex* K and its underlying space $|K|$.

[The notions of *geometric simplex* and *face* of a simplex may be used without definition.]

(b) An abstract simplicial complex has vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and simplices $\{v_1, v_2, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_5\}$ $\{v_4, v_5\}$, $\{v_4, v_6\}$ and $\{v_5, v_6\}$ and their faces. Draw a realization K of this simplicial complex as a geometric simplicial complex in $\mathbb{R}^2$.

(c) Define the *Euler characteristic* of a simplicial complex and calculate the Euler characteristic of the simplicial complex in part (b).

(d) Draw the first barycentric subdivision K' of the geometric simplicial complex K in part (b).

(e) Find the Euler characteristic of K'.

[10 marks]

A3. (a) Define what is meant by the *r-chain group* $C_r(K)$, the *r-cycle group* $Z_r(K)$, and the *r-boundary group* $B_r(K)$ of a simplicial complex K.

(b) Write down, without proof, generators for the groups $Z_1(K)$ and $B_1(K)$ of the simplicial complex K in Question A2(b). Hence find the first homology group $H_1(K)$.

[10 marks]

A4. (a) Let K be the simplicial complex $\bar{\Delta}^8$ consisting of all of the faces of the standard 8-simplex in $\mathbb{R}^9$ (including the 8-simplex itself). Write down the simplicial homology groups of K, justifying for your answer.

(b) Let L be the 3-skeleton of K. Calculate the Euler characteristic of L and hence, or otherwise, find its simplicial homology groups.

[10 marks]

SECTION B

Answer **THREE** of the FOUR questions.

If more than THREE questions are attempted then credit will be given for the best THREE answers.

B5. Let v_i be the ith standard basis vector in $\mathbb{R}^8$, $1 \leqslant i \leqslant 8$. Consider the set K of sixteen triangles with vertices v_i, v_j and v_k where ijk is one of

$$123,\ 126,\ 134,\ 148,\ 156,\ 157,\ 178,\ 237,\ 257,\ 258,\ 268,\ 346,\ 368,\ 378,\ 456,\ 458.$$

(a) Verify that K is a simplicial surface. [For the link condition, you need only check the vertices v_1 and v_8 to illustrate the method.]

(b) Represent the underlying space of $|K|$ as a polygon with edges identified in pairs, and hence represent $|K|$ by a symbol.

(c) Reduce the symbol to canonical form and hence determine the genus and orientability type of the surface $|K|$.

[15 marks]

B6. (a) Define what is meant by a *topological surface* and outline the definition of the *connected sum* $S_1 \# S_2$ of two surfaces S_1 and S_2.

(b) What is meant by a *triangulation* of a path-connected compact surface? Define the *Euler characteristic* $\chi(S)$ of a path-connected compact surface S. [You may assume that all such surfaces have a triangulation.]

(c) State and outline the proof of the relationship between $\chi(S_1)$, $\chi(S_2)$ and $\chi(S_1 \# S_2)$. Hence, calculate the Euler characteristic of the surfaces that arise in the classification theorem for compact path-connected surfaces. [You may assume the Euler characteristic of the 2-sphere, the torus and the projective plane without proof.]

(d) Explain the rôle of the Euler characteristic in proving the classification theorem.

[15 marks]

B7. (a) Describe a geometric simplicial complex K whose underlying space is homeomorphic to the *projective plane* and indicate why it has this property.

(b) Calculate the simplicial homology groups of K.

[15 marks]

B8. (a) Define what is meant by saying that two continuous functions $f_0 \colon X \to Y$ and $f_1 \colon X \to Y$ between topological spaces X and Y are *homotopic*. Prove that homotopy is an equivalence relation on the set of continuous functions from X to Y.

(b) Define what is meant by saying that two topological spaces X and Y are *homotopy equivalent*. Prove that, if X and Y are homotopy equivalent then X is path-connected if and only if Y is path-connected.

[15 marks]

SECTION C

Answer **ALL** THREE questions.

C9. (a) Let p be an odd prime. What is a *p-symmetry* of a topological surface? What is a *fixed point* of such a symmetry? Describe a p-symmetry of the projective plane with precisely one fixed point.

(b) Outline a proof that, if there is a p-symmetry of a surface S with a single fixed point, then there is a p-symmetry on $S \# P_p$ (the connected sum of S with a non-orientable surface of genus p) with a single fixed point.

[You may assume that, given a p-symmetry f on a surface S, there exists a non-empty open set U in S such that the sets $f^i(U)$, $0 \leqslant i \leqslant p-1$, are mutually disjoint.]

(c) Deduce that, if p divides $g-1$, then there is a p-symmetry of P_g with a single fixed point.

(d) Outline a proof of the converse result: if P_g has a p-symmetry with a single fixed point then p divides $g-1$.

[You may assume that, if $f\colon S \to S$ is a p-symmetry f of a closed surface S with isolated fixed points, then the identification space $S/\sim$ (where the equivalence relation is induced by $x \sim f(x)$) is also a closed surface.]

[20 marks]

C10. (a) Prove that, if a triangulation of a closed surface S with Euler characteristic χ has v vertices then

$$v \geqslant (7 + \sqrt{49 - 24\chi})/2.$$

(b) Prove that, if $v = (7 + \sqrt{49 - 24\chi})/2$, then the 1-skeleton of the triangulation gives an embedding of the complete graph on v vertices in the closed surface S.

[15 marks]

C11. (a) Define what is meant by saying that (X, A) is a *triangulable pair* of spaces.

(b) State the axioms for the reduced homology groups of triangulable spaces.

(c) Prove, from the axioms, that a homotopy equivalence of triangulable spaces $f\colon X \to Y$ induces isomorphisms $f_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$ of their reduced homology groups.

(d) Hence, show that the axioms determine the reduced homology groups of the n-sphere S^n for all $n \geqslant 0$.

[You may assume that the reduced homology groups of a point are all trivial.]

[15 marks]

END OF EXAMINATION PAPER