Spring Semester 2014–2015

MATH31072/MATH41072/MATH61072 *Algebraic Topology*

May/June Examination Solutions

General comments. *The examination went reasonably well. The marks were not scaled and there was a good proportion of first class and upper second class performances. The best scripts were really good. As usual there were a few students taking the level 4/MSc versions of the course who appeared to have done very little work on the additional reading.*

A1. (a) A *geometric simplicial surface* is a finite set K of triangles in some $\mathbb{R}^n$ satisfying the following properties.

(i) The intersection condition: Two triangles in K are either (i) disjoint, (ii) intersect in a common vertex, or (iii) intersect in a common edge.

(ii) The connectivity condition: For each pair of vertices there is a path along edges from one to the other.

(iii) The link condition: For each vertex v, the link of the vertex, i.e. the set of edges opposite v in the triangles containing v, form a simple closed polygon. [5 marks, bookwork]

(b) An *orientation* of a triangle is a cyclic ordering of the vertices. Two triangles with a common edge are *coherently oriented* if the orientations induced on the common edge are opposite. A simplicial surface is *orientable* if all of the triangles can be oriented so that each pair of triangles with a common edge are coherently oriented.

[3 marks, bookwork]

(c) The statement that this is a topological property means that, given two simplicial surfaces K_1 and K_2, if the underlying spaces $|K_1|$ and $|K_2|$ are homeomorphic, then K_1 is orientable if and only if K_2 is orientable.

[2 marks, bookwork]

[**Total: 10 marks**]

There were some rather confused definitions of orientability. It is necessary to give the little sequence of definitions. In explaining why orientability is a topological invariant it is essential to refer to the underlying spaces of the simplicial surfaces.

A2. A *geometric simplicial complex* is a non-empty finite set K of simplices in some Euclidean space $\mathbb{R}^n$ such that

(i) **the face condition:** if $\sigma \in K$ and $\tau \prec \sigma$ then $\tau \in K$;

(ii) **the intersection condition:** if σ_1 and $\sigma_2 \in K$ then $\sigma_1 \cap \sigma_2 \in K$ and $\sigma_1 \cap \sigma_2 \prec \sigma_1$, $\sigma_1 \cap \sigma_2 \prec \sigma_2$.

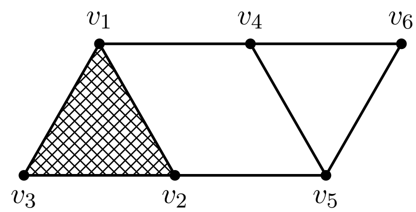
[2 marks, bookwork]

The *underlying space* $|K|$ of a simplicial complex K is given by

$$|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n$$

with the subspace topology. [1 mark, bookwork]

A realization of the given abstract complex as a geometric complex is as follows.



[2 marks, similar to question set]

The *Euler characteristic* of a simplicial complex K is given by the alternating sum

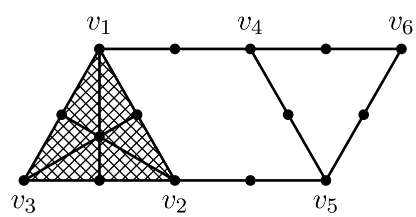
$$\chi(K) = \sum_{r=0}^{\infty} (-1)^r n_r$$

where n_r is the number of simplices of dimension r. [1 mark, bookwork]

In this case, $n_0 = 6$, $n_1 = 8$ and $n_2 = 1$ and so $\chi(K) = 6 - 8 + 1 = -1$.

[1 mark, similar to question set]

The first barycentric subdivision is as follows.



[2 marks, similar to question set]

This also has Euler characteristic -1 since the Euler characteristic is unchanged by barycentric subdivision (or because it is a topological invariant and the underlying space is unchanged) [It can also be found by counting simplices.]. [1 mark, simple application]

[**Total: 10 marks**]

This question was well done on the whole. Quite a few people forgot to define the underlying space. Some people failed to put enough extra vertices in the first barycentric subdivision.

A3. For $r \in \mathbb{Z}$. the *r-chain group of K*, denoted $C_r(K)$, is the free abelian group generated by K_r, the set of (non-empty) oriented r-simplices of K subject to the relation $\sigma + \tau = 0$ whenever σ and τ are the same simplex with the opposite orientations. [2 marks, bookwork]

For each $r \in \mathbb{Z}$ we define the *boundary homomorphism* $d_r \colon C_r(K) \to C_{r-1}(K)$ on the generators of $C_r(K)$ by

$$d_r(\langle v_0, v_1, \dots, v_r \rangle) = \sum_{i=0}^{r} (-1)^i \langle v_0, v_1, \dots, \hat{v}_i, \dots, v_r \rangle$$

and then extend linearly. Here $\hat{v}_i$ indicates that this vertex should be omitted.

[2 marks, bookwork]

The kernel of the boundary homomorphism $d_r \colon C_r(K) \to C_{r-1}(K)$ is called the *r-cycle group* of K and is denoted $Z_r(K)$. Thus

$$Z_r(K) = \{\, x \in C_r(K) \mid d_r(x) = 0 \,\}.$$

[1 mark, bookwork]

The image of the boundary homomorphism $d_{r+1}\colon C_{r+1}(K) \to C_r(K)$ is called the *r-boundary group* of K and is denoted $B_r(K)$. Thus

$$B_r(K) = \{\, x \in C_r(K) \mid x = d_{r+1}(y) \text{ for some } y \in C_{r+1}(K) \,\}.$$

[1 mark, bookwork]

In the case of K in Question A.2 we can see that

- $Z_1(K) \cong \mathbb{Z}^3$ is generated by $x_1 = \langle v_1, v_2\rangle - \langle v_1, v_3\rangle + \langle v_2, v_3\rangle$, $x_2 = \langle v_1, v_2\rangle - \langle v_1, v_4\rangle + \langle v_2, v_5\rangle - \langle v_4, v_5\rangle$ and $x_3 = \langle v_4, v_5\rangle - \langle v_4, v_6\rangle + \langle v_5, v_6\rangle$.
- $B_1(K) \cong \mathbb{Z}$ is generated by x_1.

[2 marks, similar to questions set]

The kernel of the homomorphism $Z_1(K) \to \mathbb{Z}^2$ defined by $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \mapsto (\lambda_2, \lambda_3)$ is generated by x_1 and so is $B_1(K)$. Hence by the First Isomorphism Theorem this induces an isomorphism $H_1(K) = Z_1(K)/B_1(K) \cong \mathbb{Z}^2$. [2 marks, similar to questions set]

[**Total: 10 marks**]

This was reasonably well done. Some people had trouble in just writing down the generators for the cycle and boundary group.

A4. (a) The underlying space of $K = \bar{\Delta}^8$ is the 8-simplex Δ^8 which is a convex subset of $\mathbb{R}^9$ and so is contractible. Hence it has the same homology groups as a point:

$$H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

[3 marks, standard example]

(b) For subcomplex L of K, $n_0 = 9$, $n_1 = \binom{9}{2} = 36$, $n_2 = \binom{9}{3} = 84$ and $n_3 = \binom{9}{4} = 126$ and so the Euler characteristic $\chi(L) = 9 - 36 + 84 - 126 = -69$.

[2 marks, similar to example set]

Now L is 3-dimensional and so and so has trivial homology groups in dimensions above 3. In dimensions $0 \leqslant i \leqslant 3$, $C_i(L) = C_i(K)$ with the same boundary homomorphims between these groups. Hence in dimensions $0 \leqslant i \leqslant 2$, $H_i(L) = H_i(K)$. However, in dimension 3, $B_3(L) = 0$ since $C_4(L) = 0$ and so $H_3(L) = Z_3(L)$ a free group of rank β_3, the third Betti number of L.

Now using the formula $\chi(L) = \sum_{i=0}^{3} (-1)^i \beta_i(L)$ we see that $-69 = 1 - \beta_3(L)$ (since $\beta_1(L) = \beta_2(L) = 0$) and so $\beta_3(L) = 70$. Hence

$$H_i(L) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}^{70} & \text{for } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

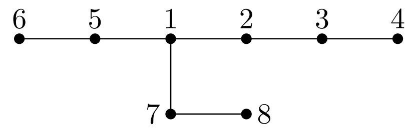
[5 marks, similar to example set]

[**Total: 10 marks**]

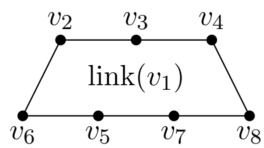
This was similar to an exercise set and a proof in the notes and so those who had studied these could on the whole do this question. Occasionally there were slips in the binomial coefficients — an 8-simplex has nine vertices.

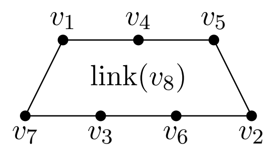
B5. The intersection condition is satisfied automatically since the vertices are linearly independent. [1 mark]

The connectivity condition is satisfied because (for example) the following edges link all of the vertices. [1 mark]



Checking the link condition for v_1 and v_8 we find the following:

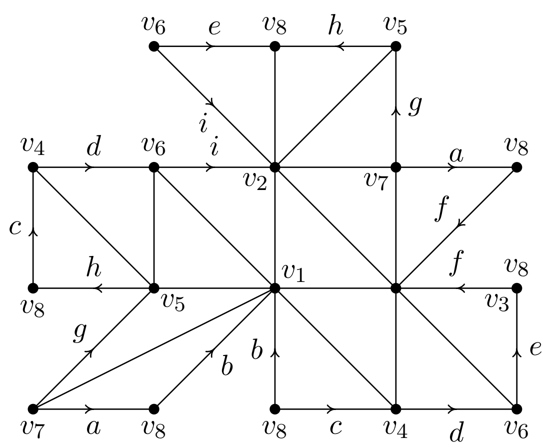




These are simple closed polygons. Hence K is a simplicial surface.

[3 marks]

(b) Now identifying edges of the triangles leads to the following polygon with edges to be identified in pairs as indicated.



This is represented by the symbol $abb^{-1}cdeff^{-1}a^{-1}ghe^{-1}ii^{-1}d^{-1}c^{-1}h^{-1}g^{-1}$.

[5 marks]

(c) Reducing this symbol to canonical form using the standard algorithm gives the following.

$$\begin{aligned}
&abb^{-1}cdeff^{-1}a^{-1}ghe^{-1}ii^{-1}d^{-1}c^{-1}h^{-1}g^{-1}\\
&\sim \quad \dot{a}cd\dot{e}(\dot{a}^{-1})(gh)\dot{e}^{-1}d^{-1}c^{-1}h^{-1}g^{-1} \quad \text{(cancelling } xx^{-1})\\
&\sim \quad \dot{a}(cd\dot{e})(gh)\dot{a}^{-1}\dot{e}^{-1}d^{-1}c^{-1}h^{-1}g^{-1} \quad (\text{since } \ldots xUVx^{-1}\ldots \sim \ldots xVUx^{-1}\ldots)\\
&\sim \quad \dot{a}ghcd(\dot{e}\dot{a}^{-1}\dot{e}^{-1})d^{-1}c^{-1}h^{-1}g^{-1} \quad (\text{since } \ldots xUVx^{-1}\ldots \sim \ldots VUx^{-1}\ldots)\\
&\sim \quad (aea^{-1}e^{-1})ghcdd^{-1}c^{-1}h^{-1}g^{-1} \quad (\text{since } xUx^{-1} \text{ commutes with other terms})\\
&\sim \quad xyx^{-1}y^{-1} \quad \text{(cancelling } xx^{-1} \text{ and relabelling).}
\end{aligned}$$

Hence the surface is orientable of genus 1 (the torus).

[5 Marks]

[Total: 15 marks, similar to questions set]

Most people did this question. The main error was in not gluing the triangles together to form a polygon. You need to attach each new triangle by an edge not just by a vertex. Some people struggled with the reduction algorithm and quite a few did not bother to give any indication of the rules they were using.

B6. (a) A topological surface is a non-empty Hausdorff second countable topological space S which is locally planar, i.e. each point $x \in X$ lies in an open subset $U \subset X$ which is homeomorphic to an open subset of the plane $\mathbb{R}^2$ with the usual topology.

Suppose that S_1 and S_2 are non-empty path-connected topological surfaces. Choose subspaces $V_1 \subset S_1$ and $V_2 \subset S_2$ which are homeomorphic to the open disc $B_1(\mathbf{0}) \subset \mathbb{R}^2$ by homeomorphisms

$$\phi_i \colon B_1(\mathbf{0}) \to V_i \quad \text{for } i = 1 \text{ and } i = 2$$

We form the connected sum $S_1 \# S_2$ by removing the interiors of smaller discs, i.e. $\phi_i\big(B^2_{1/2}(\mathbf{0})\big)$ and gluing along the boundary circles. More precisely, it is the quotient space of the disjoint union

$$S = \Big[\Big(S_1 - \phi_1\big(B^2_{1/2}(\mathbf{0})\big)\Big) \sqcup \Big(S_2 - \phi_2\big(B^2_{1/2}(\mathbf{0})\big)\Big)\Big] \Big/ \sim$$

where $\phi_1(\boldsymbol{u}) \sim \phi_2(\boldsymbol{u})$ for $\boldsymbol{u} \in B^2_1(\mathbf{0})$ with $|\boldsymbol{u}| = 1/2$.

[4 marks]

(b) A triangulation of a path-connected compact surface S is a homeomorphism $h \colon |K| \to S$ where $|K|$ is the underlying space of a simplicial surface K.

Given such a triangulation of a surface S, then the Euler characteristic of S, $\chi(S)$, is defined by $\chi(S) = v - e + f$ where v is the number of vertices in K, e is the number of edges in K and f is the number of triangles in K. This can be shown to be a topological invariant.

[2 marks]

(c) Suppose that S_1 and S_2 are two such surfaces with $|K_1| \cong S_1$ and $|K_2| \cong S_2$ then we can form K such that $|K| \cong S_1 \# S_2$ by removing a triangle from each of K_1 and K_2 and identifying the corresponding edges and vertices of these two triangles. Then $f = f_1 + f_2 - 2$ (two triangles removed), $e = e_1 + e_2 - 3$ (three pairs identified), $v = v_1 + v_2 - 3$ (three pairs identified). Thus $\chi(K) = (v_1 + v_2 - 3) - (e_1 + e_2 - 3) + (f_1 + f_2 - 2) = (v_1 - e_1 + f_1) + (v_2 - e_2 + f_2) - 2 = \chi(S_1) + \chi(S_2) - 2$.
$\chi(S^2) = 2$.

Hence, by induction on g, $\chi(T_g) = 2 - 2g$ since $\chi(T_1) = 0$ and, for $k \geqslant 1$, if the result holds for $g = k$, $\chi(T_k) = 2 - 2k$ and so $\chi(T_{k+1}) = \chi(T_k \# T_1) = (2 - 2k) + 0 - 2 = 2 - 2(k+1)$ and so the result holds for $g = k + 1$.

Similarly, $\chi(P_g) = 2 - g$ since $\chi(P_1) = 1$ and, for $k \geqslant 1$, if the result holds for $g = k$, $\chi(P_k) = 2 - k$ and so $\chi(P_{k+1}) = \chi(P_k \# P_1) = (2 - k) + 1 - 2 = 2 - (k+1)$ and so the result holds for $g = k + 1$.

[5 marks]

(d) The Euler characteristic is used in the proof of the classification theorem to help distinguish the spaces in the list.

The Euler characteristic shows that the surfaces T_g for $g \geqslant 1$ are all topologically distinct from each other and from S^2, and the surfaces P_g for $g \geqslant 1$ are all topologically distinct from each other and from S^2. However, for even numbers $2 - 2k$ ($k > 0$) there are two surfaces in the list, T_k and P_{2k}, with this Euler characteristic.

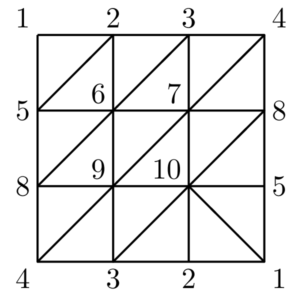
[2 marks]

[**Total: 17 marks**]

[This is a summary of coursework but requires the student to have a good overview of the first three sections of the course. The inductive proof in (c) was left as an exercise.]

This was reasonably well done although summarizing a block of material can be quite challenging. Most people didn't state the definition of the Euler characteristic in part (b) quite right since they didn't refer to the triangulation. There were some quite odd answers to part (d). Quite a few people didn't really have any idea what sort of thing I was asking for.

B7. Write v_i for the ith standard basis vector in $\mathbb{R}^{10}$, $1 \leqslant i \leqslant 9$. Let K be the set of 2-simplices $\langle v_i, v_j, v_k\rangle$ where (i,j,k) are the vertices of a triangle in the triangulation of the unit square I^2 shown below together with their faces. Then K is a simplicial complex with underlying space $|K|$ homeomorphic to the projective plane.



The intersection condition is automatic since the vertices are linearly independent vectors and the face condition is automatic by definition.

Now we can define a continuous function $f\colon I^2 \to |K|$ by mapping the point i in the unit square (in the above picture) by $i \mapsto v_i$ and extending linearly over each triangle. This is continuous by the Gluing Lemma (since the triangles are all closed subsets of I^2) and induces a continuous bijection $F\colon I^2/\!\sim\; \to |K|$ which is therefore a homeomorphism where $\sim$ is the equivalence relation given by $(s,0) \sim (s-1,1)$ and $(0,t) \sim (1,1-t)$ which is known to give the projective plane.

[5 marks, similar to bookwork]

Since K is clearly connected $H_0(K) \cong \mathbb{Z}$ and since K is 2-dimensional $H_i(K) = 0$ for $i > 2$ and $i < 0$.

To find $Z_1(K)$ notice that if $x \in Z_1(K)$ then $x \sim x'$ where x' only involves edges corresponding to the edges of the template together with three 'internal' edges, say $\langle v_5, v_6\rangle$, $\langle v_7, v_8\rangle$ and $\langle v_2, v_{10}\rangle$. Since other edges can be eliminated. For example $\langle v_2, v_5\rangle \sim \langle v_1, v_5\rangle - \langle v_1, v_2\rangle$ since $d_2\langle v_1, v_2, v_5\rangle = \langle v_2, v_5\rangle - \langle v_1, v_5\rangle + \langle v_1, v_2\rangle \sim 0$. However, since $x \in Z_1(K)$, $x' \in Z_1(K)$ and so x' cannot involve these internal edges since they have vertices which would cancel out on taking the boundary.

Considering the edges corresponding the boundary of the template we see that the cycles containing these edge are generated by

$$x = \langle v_1, v_2\rangle + \langle v_2, v_3\rangle + \langle v_3, v_4\rangle + \langle v_4, v_8\rangle - \langle v_5, v_8\rangle - \langle v_1, v_5\rangle.$$

Let V be the subgroup of $C_1(K)$ generated by x. Then $Z_1(K) = V + B_1(K)$.

Hence $H_1(K) = Z_1(K)/B_1(K) = (B_1(K) + V)/B_1(K) = V/\big(V \cap B_1(K)\big)$ by the Second Isomorphism Theorem.

If $d_2(z) \in V$ then z must be a multiple of $y = \langle v_1, v_2, v_5\rangle + \dots$ (all the 2-simplices oriented clockwise). But $d_2(y) = 2x$. Hence $V \cap B_1(K) \cong \mathbb{Z}$ generated by $2x$. Hence $H_1(K) \cong \mathbb{Z}_2$ generated by $[x]$.

For $z \in Z_2(K)$, z must be a multiple of y but since $d_2(y) \neq 0$ if follows that $Z_2(K) = 0$ and $H_2(K) = 0$.

Conclusion; $H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}_2 & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$ [10 marks, example set]

[**Total: 15 marks**]

Those who attempted this made a reasonable stab at it on the whole. It is quite difficult to know quite how much to write and I was fairly relaxed about this if people seemed to know what they were doing.

B8. (a) Two continuous functions of topological spaces $f_0\colon X \to Y$ and $f_1\colon X \to Y$ are *homotopic*, written $f_0 \simeq f_1$, if there is a continuous map $H\colon X \times I \to Y$ such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$. We call H a *homotopy* between f_0 and f_1 and write $H\colon f_0 \simeq f_1\colon X \to Y$. [2 marks, bookwork]

There are three conditions for an equivalence relation.

reflexivity: Given a continuous function $f\colon X \to Y$ then $f \simeq f$. A homotopy is given by $H(x,t) = f(x)$.

symmetry: Given homotopic functions $f_0 \simeq f_1\colon X \to Y$ then $f_1 \simeq f_0$. Given a homotopy $H\colon f_0 \simeq f_1$ then a homotopy $K\colon f_1 \simeq f_0$ is given by $K(x,t) = H(x, 1-t)$.

transitivity: Given homotopic functions $f_0 \simeq f_1\colon X \to Y$ and $f_1 \simeq f_2\colon X \to Y$ then $f_0 \simeq f_2\colon X \to Y$. Given homotopies $H\colon f_0 \simeq f_1$ and $K\colon f_1 \simeq f_2$ then a homotopy $L\colon f_0 \simeq f_2$ is given by

$$L(x,t) = \begin{cases} H(x,2t) & \text{for } 0 \leqslant t \leqslant 1/2, \\ K(x,2t-1) & \text{for } 1/2 \leqslant t \leqslant 1. \end{cases}$$

This is well-defined since $H(x,1) = f_1(x) = K(x,0)$ and is continuous by the Gluing Lemma.

Hence homotopy is an equivalence relation. [5 marks, exercise set]

(b) A continuous function $f\colon X \to Y$ is a *homotopy equivalence* when there it has a *homotopy inverse* $g\colon Y \to X$ which means that $g \circ f \simeq I_X\colon X \to X$, the identity map, and $f \circ g \simeq I_Y\colon Y \to Y$. In this case we say that X and Y are *homotopy equivalent* spaces and denote this by $X \equiv Y$ (or sometimes $X \simeq Y$). [2 marks, bookwork]

Suppose that X and Y are homotopy equivalent spaces with maps as above. Suppose that X is path-connected. To see that Y is path-connected, let y_0, $y_1 \in Y$. Then since X is path-connected there is a path $\sigma\colon [0,1] \to X$ from $g(y_0)$ to $g(y_1)$. Hence $f \circ \sigma\colon [0,1] \to Y$ is a path in Y from $f\big(g(y_0)\big)$ to $f\big(g(y_1)\big)$.

Let $H\colon f \circ g \simeq I_Y$. Then $\sigma_0(t) = H(y_0, t)$ gives a path in Y from $f\big(g(y_0)\big)$ to y_0 and $\sigma_1(t) = H(y_1, t)$ gives a path in Y from $f\big(g(y_1)\big)$ to y_1. The product of the three paths $\overline{\sigma_0}$ (reverse path), σ and σ_1 gives a path in Y from y_0 to y_1. Hence Y is path-connected.

In just the same way, reversing the roles of f and g, if Y is path-connected then so is X

[6 marks, similar to example set]

[**Total: 15 marks**]

Those who tried this question usually knew the definitions and most could show that homotopy is an equivalence relation. The problem in part (b) was on the whole not done well and only a few people managed it.

C9. (a) A *p-symmetry* of a topological surface S is a homeomorphism $f\colon S \to S$ such that $f^p = f \circ \cdots \circ f = 1$, the identity, and $f \neq 1$.

A *fixed point* of a p-symmetry $f\colon S \to S$ is a point $x \in S$ such that $f(x) = x$.

Let $f\colon S^2 \to S^2$ be a rotation about a diameter through an angle $2\pi/p$. This is a p-symmetry with two fixed points (at the ends of the diameter). The map f induces $F\colon P^2 = S^2/(x \sim \pm x) \to P^2$ with one fixed point.

[5 marks, bookwork]

(b) Let $f\colon S \to S$ be a p-symmetry with a single fixed point. Let U be an open set as in hint in the question. Since S is a surface there is a closed set $A_1 \subset U$ such that $A_1 \cong D^2$. Choose a closed set $A_2 \subset P^2$ such that $A_2 \cong D^2$. Then we can form $S' = S \# P_p$ as the connected sum of

S with p copies of P^2 by removing the interiors of the sets $f^i(A_1)$, $0 \leqslant i \leqslant p-1$, from S, taking p copies of P^2 with the interior of A_2 removed and identifying the boundary circles. Then the p-symmetry $f\colon S \to S$ extends to a p-symmetry $f'\colon S' \to S'$ which cyclically permutes the p projective planes. Since f has one fixed point so has f'.

[5 marks, problem set, similar to bookwork]

(c) We have shown that $P^2 = P_1$ has a p-symmetry with one fixed point. So, if p divides $g-1$, then $g - 1 = pr$, for some r and so $g = 1 + pr$. So applying the above result r times gives a p-symmetry on P_g with a single fixed point.

[3 marks, problem set, similar to bookwork]

(d) Suppose that $f\colon S \to S$ is a p-symmetry on a closed surface S with a single fixed point a. The we can define an equivalence relation on S by $x \sim f^i(x)$ for all $x \in S$, $i \geqslant 0$. The quotient space $S' = S/{\sim}$ is also a closed surface. In this case, under the quotient map $q\colon S \to S'$, each point of S' has precisely p preimages apart from the point $[a] = \{a\} \in S'$ which has only one preimage. Choose a triangulation $|K'| \cong S'$ so that the the point $[a] = \{a\} \in S'$ corresponds to a vertex of K'. Then using the quotient map q we can construct a simplicial surface K such that $|K| \cong S$ in such a way that the map $|K| \to |K'|$ corresponding to q maps vertices to vertices, edges to edges and triangles to triangles. Hence $v(K) = pv(K') - (p-1)$ (because of each vertex of K' corresponds to p vertices of K apart from $[a]$ which corresponds to a single vertex of K), $e(K) = pe(K')$ and $f(K) = pf(K')$. Hence $\chi(K) = p\chi(K') - (p-1)$.

Now, if $S = P_g$, $\chi(K) = 2 - g$ and so $2 - g = p(\chi(K') - 1) + 1$ which gives $g - 1 = p\big(1 - \chi(K')\big)$ so that p divides $g - 1$ as required.

[7 marks, problem set, similar to bookwork]

[**Total: 20 marks**]

C10. (a) Suppose that the triangulation has e edges and f triangles. Then we know the following.

(i) $v - e + f = \chi$ (from the definition of the Euler characteristic).

(ii) $e \leqslant v(v-1)/2$ (since the maximum number of edges has every pair of vertices joined by an edge).

(iii) $2e = 3f$ (since each triangle has three edges and each edge is an edge of two triangles).

Then $\chi = v - e + f$ (by (i)) $= v - e/3$ (by (iii)) $\geqslant v - v(v-1)/6$ (by (ii)) $= (7v - v^2)/6$. Hence $v^2 - 7v + 6\chi \geqslant 0$.

Let the roots of the equation $v^2 - 7v + 6\chi = 0$ be $v_1 < v_2$. Then $v^2 - 7v + 6\chi = (v - v_1)(v - v_2) \geqslant 0 \Leftrightarrow v \leqslant v_1$ or $v \geqslant v_2$. From the usual formula the roots are given by $\big(7 \pm \sqrt{49 - 24\chi}\big)/2$ and so $v \geqslant (7 + \sqrt{49 - 24\chi})/2$ or $v \leqslant (7 - \sqrt{49 - 24\chi})/2$.

Since $v \geqslant 3$ (a triangulation includes at least one triangle), if $v \leqslant \big(7 - \sqrt{49 - 24\chi}\big)/2$, $3 \leqslant \big(7 - \sqrt{49 - 24\chi}\big)/2$ which gives $\chi \geqslant 2$ and so $\chi = 2$ (since the Euler characteristic of a surface is at most 2. This gives $v \leqslant 3$ and so $v = 3$ which means that $e = 3$ and $f = 2$. This corresponds to two triangles with the same edges and vertices which would violate the intersection condition. So this case does not arise and we must have $v \geqslant (7 + \sqrt{49 - 24\chi})/2$, as required.

[12 marks]

(b) If $v = \big(7 - \sqrt{49 + 24\chi}\big)/2$ then $v^2 - 7v + 6\chi = 0$ and so (using the equation $\chi = v - e/3$ obtained above) $e = v(v-1)/2$ which means that there is an edge between each pair of vertices and so the 1-skeleton of the triangulation must be the complete graph on v vertices.

[3 marks]

[**Total; 15 marks**]

[**These proofs were outlined in the notes with details left as exercises.**]

C11. (a) A *triangulable pair* of spaces (X, A) is a topological space X with a subspace A such that there is a homeomorphism $h\colon X \to |K|$, the underlying space of a simplicial complex K, with $h(A) = |L|$ the underlying space of a subcomplex L of K. [1 mark, bookwork]

A reduced homology theory assigns to each non-empty triangulable space X a sequence of groups $\tilde{H}_n(X)$ (for $n \in \mathbb{Z}$) and for each continuous map of triangulable spaces $f\colon X \to Y$ a sequence of homomorphisms $f_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$ such that

(i) for continuous functions $f\colon X \to Y$ and $g\colon Y \to Z$, $g_* \circ f_* = (g \circ f)_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Z)$ for all n;

(ii) for the identity map $I\colon X \to X$, $I_* = I\colon \tilde{H}_n(X) \to \tilde{H}_n(X)$ the identity map for all n;

(iii) [homotopy axiom] for homotopic maps $f \simeq g\colon X \to Y$, $f_* = g_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$ for all n;

(iv) [exactness axiom] for any triangulable pair (X, A) there are boundary homomorphisms $\partial\colon \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A)$ for all n which fit into a long exact sequence

$$\ldots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \ldots$$

and such that for any continuous function of triangulable pairs $f\colon (X, A) \to (Y, B)$ inducing a map of quotient spaces $\bar{f}\colon X/A \to Y/B$ the following diagram commutes for all n;

$$\begin{array}{ccc} \tilde{H}_n(X/A) & \xrightarrow{\bar{f}_*} & \tilde{H}_n(Y/B) \\ \partial \downarrow & & \downarrow \partial \\ \tilde{H}_{n-1}(A) & \xrightarrow{f_*} & \tilde{H}_{n-1}(B) \end{array}$$

(v) [dimension axiom] $\tilde{H}_0(S^0) \cong \mathbb{Z}$ and $\tilde{H}_n(S^0) = 0$ for all $n \neq 0$.

[7 marks, bookwork]

(c) Suppose that $f\colon X \to Y$ is a homotopy equivalence with homotopy inverse $g\colon Y \to X$. Then

$$g_* \circ f_* = (g \circ f)_* \text{ (using (i)) } = I_* \text{ (using (iii)) } = I\colon \tilde{H}_n(X) \to \tilde{H}_n(X) \text{ (by (ii))}$$

and similarly $f_* \circ g_*\colon I\colon \tilde{H}_n(Y) \to \tilde{H}_n(X)$ so that $f_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$ is an isomorphism.

[2 marks, exercise set]

(d) Now consider the pair (D^n, S^{n-1}) for which $D^n/S^{n-1} \cong S^n$. Then the exactness axiom gives the long exact sequence

$$\ldots \to \tilde{H}_i(D^n) \xrightarrow{q_*} \tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1}) \xrightarrow{i_*} \tilde{H}_{i-1}(D^n) \ldots$$

The space D^n is contractible (homotopy equivalent to a point) and so by the above all of its homology groups are trivial. Hence from this exact sequence we see that the boundary homomorphisms

$$\partial\colon \tilde{H}_i(S^n) \to \tilde{H}_{i-1}(S^{n-1})$$

are all isomorphisms. Hence, iterating these maps and using the dimension axiom we see that

$$\tilde{H}_i(S^n) \cong \tilde{H}_{i-n}(S^0) \cong \mathbb{Z} \text{ for } i = n, 0 \text{ for } i \neq n.$$

[5 marks, exercise set]
[**Total: 15 marks**]

On the whole not much wrong was written in Part C. Some people did not appear to have studied some or even all of the material.