SECTION A

A1.

(a) Define what is meant by a *topological manifold*.

(b) Give an example to show that a quotient of a topological manifold by an equivalence relation is not necessarily a topological manifold.

(c) State the classification theorem for compact path-connected topological surfaces.

[10 marks]

Solution

(a) Let n be a non-negative integer. An *n-dimensional (topological) manifold* is a topological space X which

  (i) is Hausdorff,

  (ii) is second countable (i.e. has a countable basis), and

  (iii) is *locally Euclidean*, i.e. each point $x \in X$ lies in an open subset V in X which is homeomorphic to an open subset $U \subset \mathbb{R}^n$ (with the usual topology).

[5 marks, bookwork]

(b) One crucial property is the Hausdorff property. We need to find a manifold and an equivalence relation, such that the resulting quotient is not Hausdorff. The space $\mathbb{R} \times \{0,1\}/\sim$, with $(x,0) \sim (x,1)$ for $x \neq 0$, is not Hausdorff, but $\mathbb{R} \times \{0,1\}$ is a manifold.

Alternatively, we can find an example where the quotient is not longer locally Euclidean. For example $\mathbb{R} \times \{0,1\}/\sim$, with $\sim$ induced by $(0,0) \sim (0,1)$. The resulting space (which can be shown to be homeomorphic the union of the coordinate axes in $\mathbb{R}^2$) is not locally Euclidean around the $[(0,1),(0,0)]$

[2 marks, bookwork]

(c) Every connected compact topological surface (or *closed surface*) is homeomorphic to one and only one of:

  (i) S^2,

  (ii) T_g for some $g \geqslant 1$ (where $T_1 = S^1 \times S^1$ and $T_{g+1} = T_g \# T_1$ for $g \geqslant 1$),

  (iii) P_g for some $g \geqslant 1$ (where $P_1 = P^2$ and $P_{g+1} = P_g \# P_1$ for $g \geqslant 1$). [3 marks, bookwork]

[**Total: 10 marks**]

Quite a few people had trouble to find an example for (b). Apart from this the question was in general done well. Almost everyone remembered at least the names of the three crucial properties. However, the explanation of "locally Euclidean" was not always correct. Note, that not every open subset is homeomorphic to an open subset of Euclidean space (note, that this would imply that X itself is homeomorphic to an subset of Euclidean space). The condition states only that there exist *such an open neighbourhood U for every point $x \in X$. Some people also forgot to mention that U has to cointain x (which could be done by calling U an open* neigbourhood *of x).*

A2.

(a) Define what is meant by a *geometric simplicial complex* K.
[*The notions of geometric simplex and face of a simplex may be used without definition.*]

(b) What is the *underlying space* $|K|$ of such a simplicial complex K?

(c) An abstract simplicial complex has vertices v_1, v_2, v_3, v_4, v_5 and simplices $\{v_2, v_3, v_5\}$, $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_4\}$, $\{v_4, v_5\}$ and their faces. Draw a realisation K of this simplicial complex as a geometric simplicial complex in $\mathbb{R}^2$.

(d) Define the *Euler characteristic* of a simplicial complex and calculate the Euler characteristic of the simplicial complex in part (c).

(e) Draw the first barycentric subdivision K' of the geometric simplicial complex K in part (c).

(f) Let L be the simplicial complex that is obtained from K' by removing one triangle. Find the Euler characteristic of L.

[10 marks]

Solution

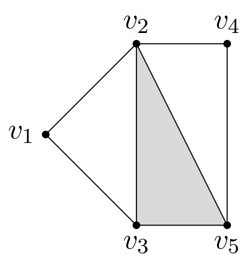
(a) A *(geometric) simplicial complex* is a non-empty finite set K of simplices in some Euclidean space $\mathbb{R}^n$ such that

(a) **the face condition:** if $\sigma \in K$ and $\tau \prec \sigma$ then $\tau \in K$,

(b) **the intersection condition:** if σ_1 and $\sigma_2 \in K$ then $\sigma_1 \cap \sigma_2 = \emptyset$ or $\sigma_1 \cap \sigma_2 \prec \sigma_1$, $\sigma_1 \cap \sigma_2 \prec \sigma_2$.

[3 marks, bookwork]

(b) A realisation is given by the following picture



[2 marks, similar to question set]

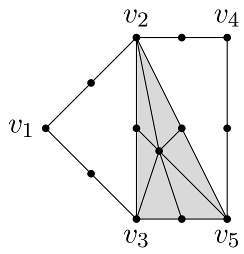
(c) The *Euler characteristic* of a simplicial complex K is given by the alternating sum

$$\chi(K) = \sum_{r=0}^{\infty} (-1)^r n_r$$

where n_r is the number of simplices of dimension r. In this case $\chi(K) = 5 - 7 + 1 = -1$.

[2 marks, bookwork]

(d) The barycentric subdivision is given by the following picture



[2 marks, similar to question set]

(e) The Euler characteristic of K' is again -1, since barycentric subdivisions does not change the Euler characteristic. Hence, $\chi(L) = \chi(K) - 1 = -2$. [1 mark, simple application]

[Total: 10 marks]

A3.

(a) Define what is meant by the *r-chain group* $C_r(K)$, the *r-cycle group* $Z_r(K)$, and the *r-boundary group* $B_r(K)$ of a simplicial complex K.

(b) Write down, without proof, generators for the groups $Z_1(K)$ and $B_1(K)$ of the simplicial complex K in Question A2(c). Hence, find the first homology group $H_1(K)$.

[10 marks]

Solution

(a) For $r \in \mathbb{Z}$. the r-chain group of K, denoted $C_r(K)$, is the free abelian group generated by K_r, the set of oriented r-simplices of K subject to the relation $\sigma + \tau = 0$ whenever σ and τ are the same simplex with the opposite orientations. [2 marks, bookwork]

For each $r \in \mathbb{Z}$ we define the boundary homomorphism $d_r : C_r(K) \to C_{r-1}(K)$ on the generators

$$d_r(\langle v_0, \ldots, v_r \rangle) = \sum_{i=0}^{r} (-1)^i \langle v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r \rangle$$

and then extend linearly. [2 marks, bookwork]

The kernel of the boundary homomorphism d_r is called the r-cycle group and denoted by $Z_r(K)$, i.e. $Z_r(K) = \{c \in C_r(K) \mid d_r(c) = 0\}$. [1 mark, bookwork]

The image of the boundary homomorphism d_{r+1} is called the r-boundary group and is denoted by $B_r(K)$, i.e. $B_r(K) = \{d_{r+1}(c) \mid c \in C_{r+1}(K)\}$. [1 mark, bookwork]

(b) $Z_1(K) \cong \mathbb{Z}^3$ is generated by

$$\begin{aligned} z_1 &= \langle v_1, v_3 \rangle + \langle v_3, v_2 \rangle + \langle v_2, v_1 \rangle \\ z_2 &= \langle v_3, v_5 \rangle + \langle v_5, v_2 \rangle + \langle v_2, v_3 \rangle \\ z_3 &= \langle v_5, v_4 \rangle + \langle v_4, v_2 \rangle + \langle v_2, v_5 \rangle \end{aligned}$$

$B_1(K) \cong \mathbb{Z}$ is generated by z_2. [2 marks, similar to question set]

We obtain

$$H_1(K) = Z_1(K)/B_1(K) = (\mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \oplus \mathbb{Z}z_3)/\mathbb{Z}z_2 \cong \mathbb{Z}^2.$$

[2 marks, similar to question set]

[Total: 10 marks]

In general the question was done well. Some of you could have saved some time for counting simplices by referring to the fact, that barycentric subdivision does not change the Euler characteristic. Also note, that the formulation of the question asks you to remove a triangle from K not from the underlying space of K. This implies all the faces of this triangle are still present in L.

A4.

(a) Consider the simplicial complex K consisting of the two 4-simplices $\langle 0, e_1, e_2, e_3, e_4 \rangle \in \mathbb{R}^4$ and $\langle e_1 + e_2 + e_3 + e_4, e_1, e_2, e_3, e_4 \rangle \in \mathbb{R}^4$ (which intersect in a 3-simplex as a common face) and all their faces. Give an argument why the homology groups of K are given by

$$H_i(K) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{else.} \end{cases}$$

(b) Let L be the 3-skeleton of K. Calculate the Euler characteristic of L and find its homology groups.

[10 marks]

Solution

(a) The underlying space is contractible. Indeed, $H(t, x) := e_1 + t \cdot (x - e_1)$ gives a homotopy between the identity and the constant map. Because of the homotopy invariance of homology the homology groups are the same as for a point. [3 marks, similar to question set]

(b) For the Euler characteristic of $|K|$ we have $\chi(|K|) = \chi(*) = 1$, since the Euler characteristic depends only on the ranks of the homology groups. On the other hand we have

$$\chi(L) = \sum_k^3 (-1)^r n_r = \chi(K) - n_4 = \chi(K) - 2 = -1,$$

since we have exactly two 4-dimensional simplices in K. [2 marks, similar to question set]

Now L is 3-dimensional and so and so has trivial homology groups in dimensions above 3. For $0 \leqslant i \leqslant 3$ we have $C_i(K) = C_i(L)$ and the boundary homomorphisms are the same. Hence, $H_i(K) = H_i(L)$ for $0 \leqslant i \leqslant 2$. Since $C_4(L) = 0$ we have $H_3(L) = Z_3(L)$ a free group of rank β_3. Now, using the identity $-1 = \chi(L) = \sum(-1)^i\beta_i$ we obtain $1 - \beta_3 = -1$(since $\beta_1 = \beta_2 = 0$) and so $\beta_3 = 2$ and $H_3(L) = \mathbb{Z}^2$. [5 marks, similar to question set]

[Total: 10 marks]

Most people realised, that the underlying space is contractible. However, I wanted also to see an argument why this is true. The easiest way to calculate the Euler characteristic was to use (a) and the Euler-Poincaré formula. However, doing the actual counting of simplices was equally good, but of course more time consuming. For getting full marks it was important to argue why $H_3(L)$ is a free (abelian) group and therefore $H_3(L) \cong \mathbb{Z}^{\beta_3}$.

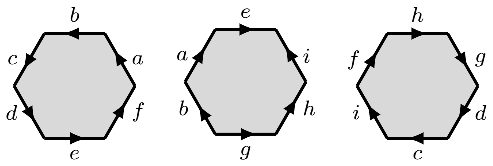
SECTION B

B5.

(a) Explain how a *surface symbol* may be used to represent a closed surface arising from the identification in pairs of the edges of a polygon.

(b) State the *classification theorem* for surface symbols.

(c) The boundaries of three discs are identified as shown below.



Find a symbol for the resulting closed surface. By reducing the symbol to canonical form, or otherwise, identify the surface up to homeomorphism.

[15 marks]

Solution

(a) To write down a symbol respresenting a topological polygon with edges identified in pairs a letter is assigned to each edge of the polygon, assigning the same letter to two edges if and only if they are identified. starting at any vertex, write down the letters in sequence going either clockwise or counterclockwise around the boundary, assigning the exponent -1 at the second appearance if the identification reverses the order of points one the corresponding pair of edges. [4 marks, bookwork]

(b) The classification theorem states that every closed surface is representatable by one and only one of the following symbols:

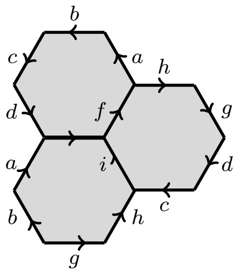
(i) xx^{-1},

(ii) $x_1y_1x_1^{-1}y_1^{-1}\dots x_gy_gx_g^{-1}y_g^{-1}$,

(iii) $x_1x_1\dots x_gx_g$.

[2 marks, bookwork]

(c) We can produce a single polygon with edges to be identified in pairs by using three edge identifications to join up the polygons as follows.



This may be represented by the symbol $abcda^{-1}b^{-1}ghc^{-1}d^{-1}g^{-1}h^{-1}$.

[3 marks, similar to question set]

Now reducing the symbol for the polygon identifications to standard form gives the following.

$$\begin{aligned} abcd(a^{-1})(b^{-1}ghc^{-1})d^{-1}g^{-1}h^{-1} &\sim a(bcd)(b^{-1}ghc^{-1})a^{-1}d^{-1}g^{-1}h^{-1} \\ &\sim ab^{-1}ghc^{-1}bc(da^{-1}d^{-1})g^{-1}h^{-1} \\ &\sim (ada^{-1}d^{-1})b^{-1}gh(c^{-1}bc)g^{-1}h^{-1} \\ &\sim (ada^{-1}d^{-1})(b^{-1}c^{-1}bc)(ghg^{-1}h^{-1}) \\ &\sim (x_1y_1x_1^{-1}y_1^{-1})(x_2y_2x_2^{-1}y_2^{-1})(x_3y_3x_3^{-1}y_3^{-1}). \end{aligned}$$

[5 marks, similar to question set]

Hence the surface is orientable of genus 3.

[1 marks, similar to question set]

[**Total: 15 marks**]

All students attempted this question and most did well up to the reduction of the symbol to normal form. Here, some people relied on ad-hoc techniques instead of using the algorithm from the lecture. Usually this took much more than five steps and often lead to confusion.

B6.

(a) Outline the definition of the *connected sum* $S_1 \# S_2$ of two connected surfaces S_1 and S_2.

(b) Define the *orientability type* of a compact path-connected surface.
[*You may assume that every such surface admits a triangulation*]

(c) Show that $S_1 \# S_2$ is orientable if S_1 and S_2 are.

(d) Show $H_2(S) \neq 0$ if S is orientable.

[15 marks]

Solution

(a) Suppose that S_1 and S_2 are non-empty path-connected topological surfaces. Choose subspaces $V_1 \subset S_1$ and $V_2 \subset S_2$ which are homeomorphic to the open disc $B_1(\mathbf{0}) \subset \mathbb{R}^2$ by homeomorphisms

$$\phi_i\colon B_1(\mathbf{0}) \to V_i \quad \text{for } i = 1 \text{ and } i = 2$$

We obtain the connected sum by removing the interiors of smaller discs, i.e. $\phi_i\big(B^2_{1/2}(\mathbf{0})\big)$ and glue along the boundary circles. More precisely, we define the quotient space of the disjoint union

$$S = \Big[\Big(S_1 - \phi_1\big(B^2_{1/2}(\mathbf{0})\big)\Big) \sqcup \Big(S_2 - \phi_2\big(B^2_{1/2}(\mathbf{0})\big)\Big)\Big] \Big/\!\sim$$

where $\phi_1(\mathbf{u}) \sim \phi_2(\mathbf{u})$ for $\mathbf{u} \in B^2_1(\mathbf{0})$ with $|\mathbf{u}| = 1/2$.

[4 marks, bookwork]

(b) An *orientation* for the the triangle is given by a cyclic ordering of the vertices. An orientation of an edge is a ordering of the vertices. Two triangles with a common edge are *coherently oriented* if the orientations induced on the common edge are opposite. An *orientation* of a simplicial surface is a choice of orientation for each triangle so that each pair of triangles with a common edge are coherently oriented. If such an orientation exists for some triangulation of a surface S, then S is called *orientable.* [4 marks, bookwork]

(c) Suppose that S_1 and S_2 are orientable and $|K_1| \cong S_1$ and $|K_2| \cong S_2$. Then we may obtain K such that $|K| \cong S_1 \# S_2$ by removing a triangle $\langle v_1, v_2, v_3\rangle$ from K_1 and a triangle $\langle v_1', v_2', v_3'\rangle$ from K_2 and identifying the vertices $v_i \sim v_i'$ and the edges between them. Chose an orientation for K_1 choosing the oriented triangle $\langle v_1, v_2, v_3\rangle$ and extending over K_1 by coherence. Similarly for K_2 starting with the oriented triangle $\langle v_3', v_2', v_1'\rangle$. The resulting orientation of the triangles of K is coherent. [4 marks, bookwork]

(d) Suppose that K is orientable. Then we may choose an orientations for all the triangles which are coherent across all the edges. Put $z =$ the sum of these oriented triangles. Then, coherence means that $d_2(z) = 0$ and so $z \in Z_2(K) = H_2(K)$ and so $H_2(K) \neq 0$. [3 marks, question set]

[Total: 15 marks]

Everyone who attempted the question had the correct idea how the connected sum is constructed. However, sometimes the description was lacking exactness and details. The same holds for the definition of the orientability type. In (b) most people scored at least partial marks. A common shortcoming was that people didn't mention that one has to make a compatible choice of the orientations for K_1 and K_2, respectively, in order to obtain an orientation for K. In (c) some people tried to prove the opposite direction of the implication, which is actually harder to show.

B7.

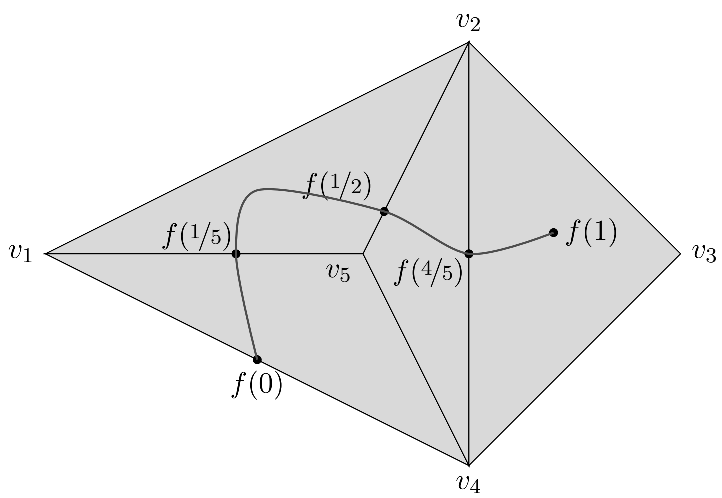
(a) Let K and L be simplicial complexes. Define what is meant by a *simplicial map* $|K| \to |L|$ (with respect to K and L). Define what is meant by a *simplicial approximation* to a continuous map $f\colon |K| \to |L|$ (with respect to K and L).

Prove that a vertex map s with $f(\mathrm{star}(v)) \subset \mathrm{star}(s(v))$ for all vertices v of K induces a simplicial approximation to f.

(b) Consider the simplicial complex L with vertices v_1, v_2, v_3, v_4, v_5, which is drawn below, and an injective continuous map $f\colon [0,1] \to |L|$ with

$$f(0) \in \langle v_1, v_4\rangle,\ f(1/5) \in \langle v_1, v_5\rangle,\ f(1/2) \in \langle v_2, v_5\rangle,\ f(4/5) \in \langle v_2, v_4\rangle,\ f(1) \in \langle v_2, v_3, v_4\rangle$$

and having the image indicated in the picture. Let K be the simplicial complex consisting just of the simplex $\langle 0, 1\rangle$ and its faces. Give a simplicial approximation to f on a sufficiently fine barycentric subdivision $K^{(m)}$ of K.



[15 marks]

Solution

(a) A map of simplicial complexes $s : K \to L$ is induced by a map of the vertex sets $s_0 : V(K) \to V(L)$ so that if $\{v_0, v_1, ..., v_r\}$ is an r-simplex of K then $\{s_0(v_0), s_0(v_1), ..., s_0(v_r)\}$ is a simplex in L (possibly of lower dimension since s_0 need not be an injection on the vertices of the simplex. Such a map of the vertices may be extended by linearity over the simplices and gives a continuous function $|s| : |K| \to |L|$ by the Gluing Lemma. A function between the underlying spaces which arises in this way is called a simplicial map.

[3 marks, bookwork]

We say that a simplicial map $|s| : |K| \to |L|$ is a simplicial approximation to a continuous map $f : |K| \to |L|$ if, for each point $x \in |K|$, the point $|s|(x)$ belongs to the carrier of $f(x)$ i.e. simplex of L whose interior contains $f(x)$.

[2 marks, bookwork]

Given a point x in the interior of $\langle v_0, \ldots, v_r\rangle$ it is contained in $\bigcap_{i=0}^r \mathrm{star}(v_i)$ and, hence,

$$f(x) \in f\left(\bigcap_{i=0}^{r} \mathrm{star}(v_i)\right) \subset \bigcap_{i=0}^{r} (\mathrm{star}(s(v_i)))$$

In particular, $\bigcap_{i=0}^{r}(\mathrm{star}(s(v_i))$ is non-empty. If the interior of σ is contained in $\bigcap_{i=0}^{r}(\mathrm{star}(s(v_i))$ then $s(v_0), \ldots, s(v_r)$ have to be vertices of σ. On the one hand this implies that $\langle s(v_0), \ldots, s(v_r)\rangle$ is a face of σ. In particular, it is a simplex in L. Hence, s is admissible. On the other hand the carrier of every point in $\bigcap_{i=0}^{r}(\mathrm{star}(s(v_i))$ contains $\langle s(v_0), \ldots, s(v_r)\rangle$ and hence $|s|(x)$.

[3 marks, question set]

(b) We have to take $K^{(2)}$ consisting of the intervals $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$ and $[3/4, 1]$ and their endpoints. Now, one observes that

$$\begin{aligned}
\mathrm{star}(0) &= [0, 1/4) \subset [0, 1/2) = f^{-1}(\mathrm{star}(v_1)),\\
\mathrm{star}(1/4) &= (0, 1/2) \subset (0, 4/5) = f^{-1}(\mathrm{star}(v_5)),\\
\mathrm{star}(1/2) &= (1/4, 3/4) \subset (0, 4/5) = f^{-1}(\mathrm{star}(v_5)),\\
\mathrm{star}(3/4) &= (1/2, 1) \subset [1/5, 1] = f^{-1}(\mathrm{star}(v_2)),\\
\mathrm{star}(1) &= (3/4, 1] \subset [1/5, 1] = f^{-1}(\mathrm{star}(v_2)).
\end{aligned}$$

Hence, by (a) the vertex map s given by $s(0) = v_1$, $s(1/4) = s(1/2) = v_5$ and $s(3/4) = s(1) = v_2$ defines a simplicial approximation to f. [7 marks, similar to question set]

[Total: 15 marks]

Problems occured already in the definition of simplicial maps and simplicial approximations. A very common mistake was to demand that $\langle v_0, \ldots, v_r\rangle$ is a simplex if and only if $\langle \phi(v_0), \ldots, \phi(v_r)\rangle$ is a simplex. But only the implication "$\Rightarrow$" is part of the definition. Indeed, note, that the constant map $\phi_w\colon V(K) \to V(L), v \mapsto w$ is an admissable vertex map, but does usually not fulfil the stronger condition above. For (b) you had to give some justification why your choice defines a simplicial approximation. Just checking the carrier condition for the vertices is not sufficient. You have to check it for every point of the underlying space $|K^{(2)}|$! It's much easier to use the condition in (a).

B8.

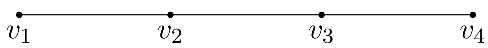
(a) Let ϕ be a simplicial map from $|K|$ to $|L|$. Show how this defines a homomorphisms

$$\phi_*\colon H_r(K) \to H_r(L).$$

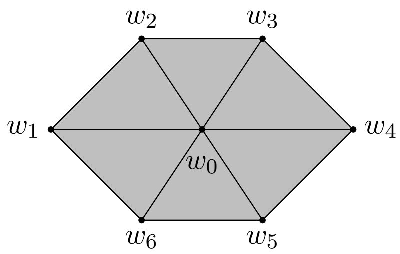
(b) Given two simplicial maps ϕ and ψ from $|K|$ to $|L|$. Define what is meant by a *chain homotopy* between ϕ and ψ.

(c) Show that the existence of a chain homotopy implies the equality $\phi_* = \psi_* : H_r(K) \to H_r(L)$ for the induced homomorphism on the level of homology.

(d) Consider the simplicial complexes K



and L



Given the simplicial maps ϕ and ψ from $|K|$ to $|L|$ defined by

$$\phi(v_1) = w_1, \phi(v_2) = w_2, \phi(v_3) = w_3, \phi(v_4) = w_4$$

and

$$\psi(v_1) = w_1, \psi(v_2) = w_6, \psi(v_3) = w_5, \psi(v_4) = w_4.$$

Explain why a chain homotopy exists between ϕ and ψ, and describe such a chain homotopy explicitely.

[15 marks]

Solution

(a) One defines $\phi_r : C_r(K) \to C_r(L)$ as follows $\phi_r(\langle v_0, \dots, v_r\rangle) = \langle \phi(v_0), \dots \phi(v_r)\rangle$ if the result is an r-simplex and 0 else. [1 mark, bookwork]

To see that this induces a homomorphism $H_r(K) \to H_r(L)$ one needs to check, that cycles are send to cycles and boundaries are send to boundaries. This follows directly from the fact that $d_r \circ \phi_r = \phi_{r-1} \circ d_r$. Indeed,

$$\begin{aligned} \phi_{r-1}(d_r(\langle v_0, \dots v_r\rangle) &= \sum_i (-1)^i \phi_{r-1}(\langle v_0, \dots, \hat{v}_i, \dots, v_r\rangle) \\ &= \sum_i (-1)^i \langle \phi(v_0), \dots, \widehat{\phi(v_i)}, \dots, \phi(v_r)\rangle \\ &= d_r(\phi_r(\langle v_0, \dots v_r\rangle)) \end{aligned}$$

obviously holds whenever $\phi_r(\langle v_0, \dots v_r\rangle) \neq 0$. However, if we have $\phi_r(\langle v_0, \dots v_r\rangle) = 0$, then $\#\{\phi(v_0), \dots, \phi(v_r)\} < r+1$. If $\#\{\phi(v_0), \dots, \phi(v_r)\} < r$ then $\#\{\phi(v_0), \dots, \widehat{\phi(v_i)}, \dots, \phi(v_r)\} < r$. Hence, every summand in $\sum_i (-1)^i \phi_r(\langle v_0, \dots, \hat{v}_i, \dots, v_r\rangle)$ vanishes. It remains to check the case when $\#\{\phi(v_0), \dots, \phi(v_r)\} = r$. Then exactly two vertices are identified with each other. W.l.o.g. we may assume these are v_0 amd v_1. Then we have

$$\phi(\langle v_0, \hat{v}_1, \dots, v_r\rangle) = \langle \phi(v_0), \widehat{\phi(v_1)}, \dots, \phi(v_r)\rangle = \langle \widehat{\phi(v_0)}, \phi(v_1), \dots, \phi(v_r)\rangle = \phi(\langle \hat{v}_0, v_1, \dots, v_r\rangle)$$

and all other summands of $\sum_i (-1)^i \phi_r(\langle v_0, \dots, \hat{v}_i, \dots, v_r\rangle)$ vanish.

[4 mark, bookwork]

(b) A chain homotopy between ϕ and ψ is a sequence of group homomorphisms $h_i : C_i(K) \to C_{i+1}(L)$ such that $\phi_i - \psi_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$. [1 mark, bookwork]

(c) Consider an element $x \in Z_i(K)$, i.e. $d_i(x) = 0$ then

$$\phi_i(x) - \psi_i(x) = d_{i+1}(h_i(x)) + h_{i-1}(d_i(x)) = d_{i+1}(h_i(x)) + 0 = d_{i+1}(h_i(x)) \in B_i(L).$$

Hence, $\phi_*(x) - \psi_*(x) = 0$ in $H_i(L)$. [3 mark, bookwork]

(d) It's easy to see, that for every $\langle v_i, v_{i+1}\rangle$ there is an acyclic subcomplex of L supporting both $\phi(\langle v_i, v_{i+1}\rangle)$ and $\psi(\langle v_i, v_{i+1}\rangle)$. For example for $i = 2$ such an acyclic subcomplex is formed by the triangles $\langle w_2, w_3, w_0\rangle$ and $\langle w_0, w_5, w_6\rangle$ and their faces. Then the proposition on acyclic support implies the existence of a chain homotopy. [2 mark, new example]

To construct a chain homotopy one starts with paths connecting $\phi(\langle v_i)$ and $\psi(\langle v_i)$. These are used to construct the elements $h_0(\langle v_i) \in H_1(L)$:

$$\begin{aligned} h_0(\langle v_1\rangle) &= h_0(\langle v_4\rangle) = 0, \\ h_0(\langle v_2\rangle) &= \langle w_2, w_0\rangle + \langle w_0, w_6\rangle, \\ h_0(\langle v_3\rangle) &= \langle w_3, w_0\rangle + \langle w_0, w_5\rangle, \end{aligned}$$

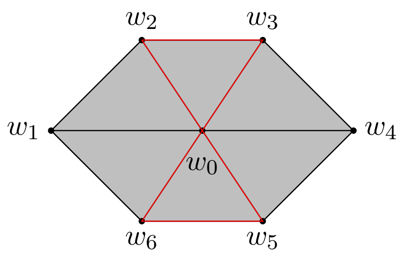
Now, we compute

$$h_0(d(\langle v_i, v_{i+1}\rangle)) + \psi(\langle v_i, v_{i+1}\rangle) - \phi(\langle v_i, v_{i+1}\rangle).$$

Which is always a cycle. For example for $i = 2$ we obtain

$$h_0(d(\langle v_2, v_3\rangle)) + \psi(\langle v_2, v_3\rangle) - \phi(\langle v_2, v_3\rangle) = -\langle w_2, w_0\rangle - \langle w_0, w_6\rangle + \langle w_3, w_0\rangle + \langle w_0, w_5\rangle + \langle w_2, w_3\rangle - \langle w_6, w_5\rangle.$$

The support of this cycle is shown in the following picture.



Since the condition for acyclic support is fulfilled we know that this cycle is also a boundary. Hence, it is possible to find a 2-chain z with $d(z) = h_0(d(\langle v_i, v_{i+1}\rangle)) + \psi(\langle v_i, v_{i+1}\rangle) - \phi(\langle v_i, v_{i+1}\rangle)$. For $i = 2$ this chain can be in princple read off the above picture. Indeed, the 2-chain is given by $z = \langle w_2, w_3, w_0\rangle + \langle w_0, w_5, w_6\rangle$. Now we use this z as image of $\langle v_i, v_{i+1}\rangle$ under h_1. In this way one obtains a chain homotopy as follows.

$$\begin{aligned} h_0(\langle v_1\rangle) &= h_0(\langle v_4\rangle) = 0, \\ h_0(\langle v_2\rangle) &= \langle w_2, w_0\rangle + \langle w_0, w_6\rangle, \\ h_0(\langle v_3\rangle) &= \langle w_3, w_0\rangle + \langle w_0, w_5\rangle, \\ h_1(\langle v_1, v_2\rangle) &= \langle w_1, w_2, w_0\rangle + \langle w_0, w_6, w_1\rangle, \\ h_1(\langle v_2, v_3\rangle) &= \langle w_2, w_3, w_0\rangle + \langle w_0, w_5, w_6\rangle, \\ h_1(\langle v_3, v_4\rangle) &= \langle w_3, w_4, w_0\rangle + \langle w_0, w_4, w_5\rangle. \end{aligned}$$

[4 mark, new example]

Only a few people attempted this question. There was some confusion about the role of $L(\sigma)$ and $h_i(\sigma)$. Although there is a connection it is not the same kind object. Indeed, $L(\sigma)$ is a simplicial complex (a subcomplex of L), but $h_i(\sigma)$ is an $(r+1)$-chain. The proof of the proposition on acyclic support constructs $h_i(\sigma)$, such that it is supported inside of $L(\sigma)$.

B9.

(a) Define what is meant by the *degree* $\deg(f)$ of a continuous selfmap $f\colon S^n \to S^n$ of a sphere. Show that your definition does not depend on the choice of a triangulation.

(b) Calculate the degree of the map $f\colon S^1 \to S^1$ given by $f(z) = z^{-1}$ for $z \in S^1 \subset \mathbb{C}$.

(c) Given two continuous maps $f, g\colon S^n \to S^n$ show that $\deg(f \circ g) = \deg(f) \cdot \deg(g)$ holds.

(d) Show that a homeomorphism $f\colon S^n \to S^n$ has either degree 1 or -1.

[15 marks]

Solution

(a) Consider a triangulation $h\colon |K| \to S^n$ of the of the n-sphere. Then $f\colon S^n \to S^n$ induces an map $(h^{-1} \circ f \circ h)\colon |K| \to |K|$. Hence, a homomorphism $(h^{-1} \circ f \circ h)_*\colon H^n(K) \to H^n(K)$. Since $H^n(K) \cong \mathbb{Z}$ the homomorphism $(h^{-1} \circ f \circ h)_*$ must be given by $[z] \mapsto \lambda[z]$ for some $\lambda \in \mathbb{Z}$. We then define $\deg(f) = \lambda$. [3 mark, bookwork]

Assume, there is a second triangulation $k\colon |L| \to S^n$. Then we have a homeomorphism $g = h^{-1} \circ k\colon |L| \to |K|$. With this definition we obtain

$$(k^{-1} \circ f \circ k)_*([z]) = (g^{-1} \circ h^{-1} \circ f \circ h \circ g)_*([z]) = g_*^{-1}((h^{-1} \circ f \circ h)_*(g_*([z]))) = g_*^{-1}(\lambda g_*([z])) = \lambda[z],$$

where the equalities follow from the definition of g, the functoriality of homology, the definition of λ above and the fact that g_*^{-1} is a homomorphism. [3 mark, bookwork]

(b) We consider the simplicial complex K in $\mathbb{C}$ formed by 1, i, -1 and $-i$ as vertices and the edges $\langle 1, i\rangle$, $\langle i, -1\rangle$, $\langle -1, -i\rangle$, $\langle -i, 1\rangle$. Via radial projection this gives a triangulation of S^1. A simplicial approximation is given by $1 \mapsto 1$, $i \mapsto -i$, $-1 \mapsto -1$ and $-i \mapsto i$. Note, that a generator of $H^n(K)$ is given by

$$z = [\langle 1, i\rangle + \langle i, -1\rangle + \langle -1, -i\rangle + \langle -i, 1\rangle]$$

Now,

$$f_*([z]) = [\langle 1, -i\rangle + \langle -i, -1\rangle + \langle -1, i\rangle + \langle i, 1\rangle] = -[z]$$

Hence, $\deg(f) = -1$.

[4 mark, similar to question set]

(c) $\deg(f \circ g)$ is given by

$$(f \circ g)_*([z]) = f_*(g_*([z])) = f_*(\deg(g)[z]) = \deg(g) \cdot (f_*([z])) = \deg(g)\deg(f)[z].$$

I.e. $\deg(f \circ g) = \deg f \cdot \deg g$. Note, that the equalities above follow from the functoriality of homology, the definition of $\deg(g)$, the fact that f_* is a group homomorphism and the definition of $\deg(f)$. [3 mark, new]

(d) Consider $\mathrm{id}_{S^n} = f \circ f^{-1}$. By the above and the fact that $\deg(\mathrm{id}) = 1$ one obtains $1 = \deg(f) \cdot \deg(f^{-1})$. But 1 and -1 are the only elements of $\mathbb{Z}$ with multiplicative inverses.

[2 mark, new]

[**Total: 15 marks**]

Again, only a few people attempted this question. They struggled to find a simplicial approximation for the map in (b).

B10.

(a) Define what is meant by saying that two continuous functions $f_0 \colon X \to Y$ and $f_1 \colon X \to Y$ between topological spaces X and Y are *homotopic*. Prove that homotopy is an equivalence relation on the set of continuous functions from X to Y .

(b) Define what is meant by saying that two topological spaces X and Y are *homotopy equivalent*. Prove that, if X and Y are homotopy equivalent, then X is path-connected if and only if Y is path-connected.

[15 marks]

Solution

(a) Two continuous functions of topological spaces $f_0 \colon X \to Y$ and $f_1 \colon X \to Y$ are homotopic , written $f_0 \simeq f_1$, if there is a continuous map $H \colon X \times I \to Y$ such that $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$. We call H a homotopy between f_0 and f_1 and write $H \colon f_0 \simeq f_1 \colon X \to Y$.

[2 mark, bookwork]

reflexivity A homotopy for $f \simeq f$ is given by $H(x, t) = f(x)$.

symmetry If $H \colon f_0 \simeq f_1 \colon X \to Y$, then $H' \colon f_1 \simeq f_0 \colon X \to Y$, with $H'(x, t) := H(x, 1 - t)$.

transitivity Assume $H \colon f_0 \simeq f_1 \colon X \to Y$ and $K \colon f_1 \simeq f_2 \colon X \to Y$. Then $L \colon f_0 \simeq f_2 \colon X \to Y$ with

$$L(x,t) \begin{cases} H(x, 2t) & 0 \leqslant t \leqslant 1/2 \\ H(x, 2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases},$$

which is well-defined and continuous by the Gluing Lemma.

[6 mark, bookwork]

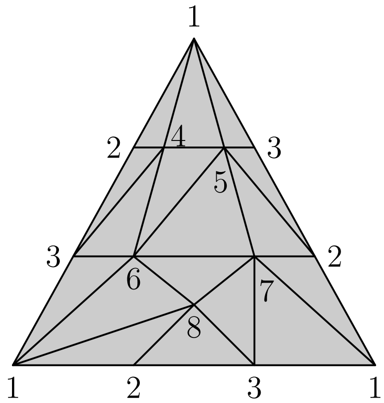
(b) A continuous function $f \colon X \to Y$ is a homotopy equivalence when there is homotopy inverse $g \colon Y \to X$ which means that $g \circ f \simeq \mathrm{id}_X \colon X \to X$ and $f \circ g \simeq \mathrm{id}_Y \colon Y \to Y$. In this case we say that X and Y are homotopy equivalent spaces and denote this by $X \equiv Y$.

Suppose that X and Y are homotopy equivalent spaces with maps as above. Suppose that X is path-connected. To see that Y is path-connected, let $y_0, y_1 \in Y$. Then since X is path-connected there is a path $\sigma \colon [0, 1] \to X$ from $g(y_0)$ to $g(y_1)$. Hence $f \circ \sigma \colon [0, 1] \to Y$ is a path in Y from $f(g(y_0))$ to $f(g(y_1))$. Let $H \colon f \circ g \simeq \mathrm{id}_Y$. Then $\sigma_0(t) = H(y_0, t)$ gives a path in Y from $f(g(y_0))$ to y_0 and $\sigma_1(t) = H(y_1, t)$ gives a path in Y from $f(g(y_1))$ to y_1. The product of the three paths $\bar{\sigma}_0$ (reverse path), σ and σ_1 gives a path in Y from y_0 to y_1 . Hence Y is path-connected. In just the same way, reversing the roles of f and g , if Y is path-connected then so is X. [7 mark, bookwork]

[**Total: 15 marks**]

Most people attempted this question and the first part was generally done well. In (b) quite a few people tried to used the fact that g is an actual inverse for f. However, this is not true in general, as g is only a homotopy inverse. One solution used the homotopy invariance of homology and concluded that $H_0(X) = \mathbb{Z} \Leftrightarrow H_0(Y) = \mathbb{Z}$. *Hence, by a result from the problem sheets also X is path-connected if and only if Y is path-connected. This was a correct argument and I awarded full marks for it.*

B11. Consider the simplicial complex in $\mathbb{R}^8$ with vertices $e_1, \ldots, e_8$ forming simplices according to the following picture:



(a) Calculate the simplicial homology groups of the simplicial complex above. You may use the fact, that every 1-cycle is homologous to one involving only edges on the boundary of the template and e.g. the following "internal" edges: $\langle 2,4\rangle$, $\langle 3,5\rangle$, $\langle 3,6\rangle$, $\langle 2,7\rangle$ and $\langle 1,8\rangle$.

(b) Use the classification theorem to show that the underlying topological space is *not* a closed surface.

[15 marks]

Solution

(a) To find $Z_1(K)$ first note, that for every 1-cycle x one has $x \sim x'$ for some x' only involving edges corresponding to edges on the boundary of the template and e.g. the following "internal" edges $\langle 3,4\rangle$, $\langle 3,5\rangle$, $\langle 2,6\rangle$, $\langle 2,7\rangle$ and $\langle 1,8\rangle$. Since all other edges can be eliminated via boundaries of triangles. However, since $x \in Z_1(K)$ we also have $x' \in Z_1(K)$ and so x' can not involve these internal edges, since their "internal" vertices wouldn't cancel out when taking the boundary.

Consider a cycle

$$x' = \lambda_1\langle 1,3\rangle + \lambda_2\langle 3,2\rangle + \lambda_3\langle 2,1\rangle.$$

Now, $d(x') = 0$ implies $\lambda_1 = \lambda_2 = \lambda_3$. Hence, the subgroup V of cycles involving only edges on the boundary of the template is generated by

$$x = \langle 1,3\rangle + \langle 3,2\rangle + \langle 2,1\rangle.$$

We have $Z_1(K) = V + B_1(K)$. It remains to determine $V \cap B_1(K)$. Consider, some non-trivial cycle $Z_2(K)$. For the inner edges of the template to cancel out when taking the boundary the cycle has to be a multiple of the sum over all triangles (with compatible orientation, e.g. all clockwise orientented), which we denote by y. But then

$$d(\ell y) = \ell \cdot 3 \cdot (\langle 1,3\rangle + \langle 3,2\rangle + \langle 2,1\rangle) = \ell \cdot 3x.$$

Hence, $V \cap B_1(K) = 3V$ and $H_1(K) = Z_1(K)/B_1(K) \cong V/(B_1 \cap V) \cong \mathbb{Z}/3\mathbb{Z}$.

For $z \in Z_2(V)$ it must be a multiple of y, but since $d(y) \neq 0$ we have $H_2(K) = Z_2(K) = 0$.

[12 marks, similar to question set]

(b) For the Euler characteristic we obtain

$$\chi = \beta_0 - \beta_1 + \beta_2 = 1 - 0 + 0 = 1.$$

From the classification theorem we see that the only possible closed surface is the projective plane. On the other hand, we have $H_1(\mathbb{P}^2) = \mathbb{Z}/2\mathbb{Z} \neq \mathbb{Z}/3\mathbb{Z} = H_1(K)$.

[3 marks, new]

[Total: 15 marks] *This was the standard calculation as demonstrated in the lectures and practiced on the problem sheets. Most people who attempted the question found the correct solution. Two serious mistakes were the following kind of arguments:*

- *Since $C_2(K) \cong \mathbb{Z}^n$, we have $B_1(K) \cong \mathbb{Z}^n$.*
  This is not *true as d_r is usually not injective!*
- *$Z_1(K) \cong \mathbb{Z}^2$ and $B_1(K) \cong \mathbb{Z}$. Hence $H_1(K) \cong \mathbb{Z}$.*
  This is not true *as $H_1(K)$ could be of the form $\mathbb{Z} \times (\mathbb{Z}/k\mathbb{Z})$ as well, where every $k > 0$ is possible. Indeed, the quotient G/H of a group G by a normal subgroup $H \subset G$ is a construction which heavily depends on the embedding of the group H inside the group G. Hence, knowing H up to isomorphy doesn't help much. We really need to understand H as a subgroup of G.*

END OF EXAMINATION PAPER