MATH41071/MATH61071 *Algebraic topology*

§2. Simplicial Surfaces

2.1 Definition. Suppose that v_0, v_1, $v_2 \in \mathbb{R}^n$ are non-colinear points. Then the *triangle* $\langle v_0, v_1, v_2\rangle$ with *vertices* v_0, v_1, v_2 is the set

$$\langle v_0, v_1, v_2\rangle = \{\, t_0v_0 + t_1v_1 + t_2v_2 \mid t_i \geqslant 0,\ t_0 + t_1 + t_2 = 1\,\}.$$

The *edges* of the triangle are $\langle v_0, v_1\rangle = \{\, t_0v_0 + t_1v_1 \mid t_i \geqslant 0,\ t_0 + t_1 = 1\,\}$, $\langle v_0, v_2\rangle$ and $\langle v_1, v_2\rangle$.
The set of points in the triangle not lying on an edge is called the *interior* of the triangle.

2.2 Proposition. Given two triangles $\langle v_0, v_1, v_2\rangle$ and $\langle v_0', v_1', v_2'\rangle$ then the bijection of the vertices $v_i \mapsto v_i'$ induces a homeomorphism between the triangle *by linear extension*:

$$\sum_{i=0}^{2} t_iv_i \mapsto \sum_{i=0}^{2} t_iv_i'.$$

2.3 Definition. A *(geometric) simplicial surface* is a finite set of triangles in some $\mathbb{R}^n$ satisfying:

(a) **the intersection condition:** the intersection of each pair of triangles in K is

  (i) empty, or
  (ii) a vertex of each of the triangles, or
  (iii) an edge of each of the triangles;

(b) **the connectivity condition:** for each pair of vertices of triangles in K, there is a path along edges from one vertex to the other;

(c) **the link condition:** for each vertex v of a triangle in K, the *link* of v, $\mathrm{link}(v)$, the set of edges opposite v in the triangles of K having v as a vertex, is a simple closed polygon.

2.4 Definition. Given a simplicial surface K in $\mathbb{R}^n$, the *underlying space* $|K|$ of K is the set of points in $\mathbb{R}^n$ which belong to some triangle of K with the usual topology as a subset of $\mathbb{R}^n$.

2.5 Proposition. If K is a simplicial surface then $|K|$ is a path-connected compact topological surface (a closed surface).

Proof. $|K|$ is a finite union of compact sets in $\mathbb{R}^n$ and so is compact. It is a subspace of $\mathbb{R}^n$ and so is Hausdorff and second countable.
The connectivity condition on K guarantees that $|K|$ is path-connected: the triangles are path-connected since they are convex subsets of $\mathbb{R}^n$ and so every point can be connected to the vertices in its triangle by a straight line; the connectivity condition shows that there is a path between each pair of vertices.
Finally, $|K|$ is locally Euclidean since each point of $|K|$ lies in an open set in $|K|$ homeomorphic to an open set in $\mathbb{R}^2$. To see this observe that there are three sorts of point:

(i) the interior points of a triangle: the interior of the triangle provides the open set;

(ii) the interior point of an edge: observe that the link condition implies that each edge lies in precisely two triangles [Exercise] and so that interior of these two triangles together with the interior of the edge forms the required open neighbourhood;

(iii) vertices: the link condition implies that the union of the triangles containing the vertex is homeomorphic to a closed disc and so the corresponding open disc provides the required open neighbourhood.

□

2.6 Definition. A *triangulation* of a closed surface S is a homeomorphism $h\colon |K| \to S$ where K is a simplicial surface.

2.7 Theorem. [T. Rado, 1925] Every closed surface has a triangulation.

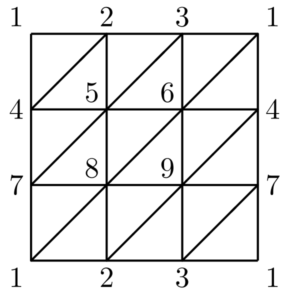
Proof. Omitted. □

2.8 Examples. (a) Write $v_0 = (0,0,0)$, $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,0,1)$. Let

$$K = \{\, \langle v_0, v_1, v_2\rangle, \langle v_0, v_1, v_3\rangle, \langle v_0, v_2, v_3\rangle, \langle v_1, v_2, v_3\rangle \,\}.$$

Then K is a simplicial surface and $|K| \cong S^2$ by radial projection from an interior point of the tetrahedron $|K|$.

(b) Now write v_i for the ith standard basis vector in $\mathbb{R}^9$, $1 \leqslant i \leqslant 9$. Let K be the set of triangles $\langle v_i, v_j, v_k\rangle$ where (i,j,k) are the vertex labels of a triangle in the triangulation of the unit square I^2 shown below. Then K is a simplicial surface with underlying space $|K|$ homeomorphic to the torus.



First of all notice that the intersection condition is automatic since the vertices are linearly independent vectors. For example

$$\langle v_1, v_2, v_3\rangle = \{\, (t_1, t_2, t_3, 0, 0, 0, 0, 0, 0) \mid t_i \geqslant 0,\ \sum t_i = 1\,\};$$

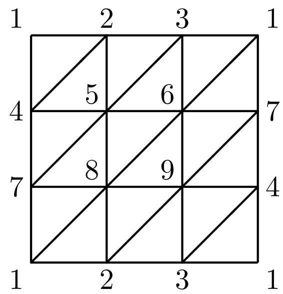
$$\langle v_2, v_3, v_8\rangle = \{\, (0, t_2, t_3, 0, 0, 0, 0, t_8, 0) \mid t_i \geqslant 0,\ \sum t_i = 1\,\}$$

and so

$$\langle v_1, v_2, v_3\rangle \cap \langle v_2, v_3, v_9\rangle = \{\, (0, t_2, t_3, 0, 0, 0, 0, 0, 0) \mid t_i \geqslant 0,\ \sum t_i = 1\,\} = \langle v_2, v_3\rangle.$$

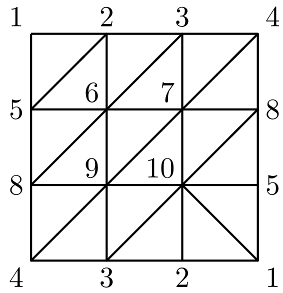
So two triangles with two common vertices intersect in their common edge, two triangles with one common vertex intersect in this vertex and two triangles with no common vertices do not intersect.
The connectivity condition is obvious from the picture.
The link condition can be checked for each vertex.
Now we can define a continuous function $f\colon I^2 \to |K|$ by mapping the point i in the unit square (in the above picture) by $i \mapsto v_i$ and extending linearly over each triangle. This is continuous by the Gluing Lemma (since the triangles are all closed subsets of I^2) and induces a continuous bijection $F\colon I^2/{\sim} \to |K|$ which is therefore a homeomorphism where $\sim$ is the equivalence relation given by $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, t)$ so that $I^2/{\sim} \cong S^1 \times S^1$.

(c) Similarly the following diagram gives a template for a triangulation of the Klein bottle.



Recall (Problems 1, Question 6) that the Klein Bottle is given by $I^2/\sim$ where $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, 1 - t)$.

(d) The following diagram gives a template for a triagulation of the projective plane.



Recall from the proof of Proposition 1.20 that the projective plane is homeomorphic to $D^2/{\sim}$ where $\mathbf{x} \sim \mathbf{x}'$ if and only if $\mathbf{x} = \mathbf{x}'$ or $\mathbf{x}' = -\mathbf{x} \in S^1 \subset D^2$. By radial projection from the centre of the disc and the square $D^2 \cong I^2$ and under this homeomorphism the equivalence relation on D^2 corresponds to the relation on I^2 given by $(s, 0) \sim (1-s, 1)$ and $(0, t) \sim (1, 1-t)$.

2.9 Remarks. Suppose that K_1 and K_2 are simplicial surfaces such that there is a bijection f from the vertices of K_1 to the vertices of K_2 such that

$$\langle v_0, v_1, v_2\rangle \text{ is a triangle in } K_1 \Leftrightarrow \langle f(v_0), f(v_1), f(v_2)\rangle \text{ is a triangle in } K_2.$$

Then f induces a homeomorphism

$$|f|\colon |K_1| \to |K_2|$$

by linear extension of f over each triangle. [$|f|$ is continuous on each triangle and so on $|K_1|$ by the Gluing Lemma and its inverse is given by $|f^{-1}|$.]
This means that given a geometric simplicial surface K, then the set of vertices $V(K)$ and the list of which triples are the vertices of a triangle determine the underlying space of K up to homeomorphism. This leads to the following definition.

2.10 Definition. An *(abstract) simplicial surface* K is a finite set $V = \{v_1, v_2, \dots, v_k\}$ (the *vertices*) together with a collection of subsets of order 3 (the *triangles*) which satisfies the connectivity and link conditions as in Definition 2.3.
An *isomorphism* $f\colon K_1 \to K_2$ of two abstract simplicial surfaces is a bijection $f\colon V_1 \to V_2$ of the two vertex sets such that

$$\langle v_0, v_1, v_2\rangle \text{ is a triangle in } K_1 \Leftrightarrow \langle f(v_0), f(v_1), f(v_2)\rangle \text{ is a triangle in } K_2.$$

2.11 Remarks. An abstract simplicial surface K with $V(K) = \{\, v_i \mid 1 \leqslant i \leqslant k \,\}$ can be realised as a geometric simplicial surface in $\mathbb{R}^k$ by putting $v_i = \varepsilon_i$ the i'th standard basis vector. As in Example 2.8(b) the intersection condition is automatic. By the above remarks (2.9) any two geometric simplicial surfaces corresponding to K will have a homeomorphic underlying

spaces. Thus the *underlying space* of an abstract simplicial surface $|K|$ is determined up to homeomorphism.
Thus the homeomorphism type of a topological surface S is completely determined by the combinatorial information in an abstract simplicial surface K such that $|K| \cong S$. This leads to two questions.

(i) What information about an abstract simplicial surface K is relevant to the topology of $|K|$?

(ii) Given two simplicial surfaces K_1 and K_2, when is it true that $|K_1| \cong |K_1|$?

Representing a simplicial surface by a symbol

2.12 Definition. A *topological polygon* (or n-gon) is a topological space which is homeomorphic to the closed disc D^2 with n points v_1, v_2, $\dots v_n$ (*vertices*) on the 'boundary' (i.e. the subspace corresponding to $S^1 \subset D^2$ — this can be shown to be well-defined not depending on the choice of homeomorphism). The vertices are labelled so that v_i and v_{i+1} are adjacent for all i (modulo n). The arcs v_1v_2, v_2v_3, $\dots v_nv_1$ are the *edges* of the polygon.

2.13 Example. Let

$$K = \{\, \langle v_i, v_j, v_k \rangle \mid ijk = 125, 126, 134, 136, 145, 234, 235, 246, 356, 456 \,\}.$$

We can take the ten triangles and (starting from $\langle v_1, v_2, v_5 \rangle$) attach then one at a time by an edge according to the labelling in K. This leads to a topological polygon with twelve edges. The labelling indicates a pairing of the edges. Using the argument of Example 2.8(b) we can see that identifying the edges of this polygon in pairs according to the labelling gives an identification space which is homeomorphic to the the underlying space $|K|$.

2.14 Proposition. Given a simplicial surface K, its underlying space is homeomorphic to the identification space formed by identifying the edges of a topological polygon in pairs.

Outline proof. Given a simplicial surface K with n triangles we can show by induction and the connectivity and link conditions that the triangles can be labelled σ_1, σ_2, $\dots$, σ_n so that each triangle σ_i has at least one edge e_i in common with one of the triangles $\sigma_1, \dots \sigma_{i-1}$ for $1 < i \leqslant n$.
Assume this is not possible. Then after step i we don't find a triangle in $K \setminus \{\sigma_1, \dots, \sigma_{i-1}\}$ having an edge in common with one of the triangles σ_1, $\dots$, σ_{i-1}. Hence, the edge sets are disjoint. On the other hand because of the connectivity condition there must be a triangle in $K \setminus \{\sigma_1, \dots, \sigma_{i-1}\}$ having

a vertex v in common with on of the triangles $\sigma_1, \dots \sigma_{i-1}$. By the link condition there must be a vertex w in the link with one ingoing edge $\langle v', w\rangle$ being part of $\{\sigma_1, \dots, \sigma_{i-1}\}$ and the other one $\langle v'', w\rangle$ being in $K \setminus \{\sigma_1, \dots, \sigma_{i-1}\}$, else the link would be disconnected. Hence, the corresponding triangles $\langle v, v', w\rangle$ and $\langle v, v'', w\rangle$ lie in $\{\sigma_1, \dots, \sigma_{i-1}\}$ and $K \setminus \{\sigma_1, \dots, \sigma_{i-1}\}$, respectively, and share an edge. A contradiction.
Having done this we choose a σ_i' for each i, $1 \leqslant i \leqslant n$ such that the σ_i' are all disjoint with linear homeomorphisms $\phi_i \colon \sigma_i' \to \sigma_i$. Then there is a continuous map

$$\phi\colon \bigsqcup_{i=1}^{n} \sigma_i' \to |K|$$

defined by $\phi|\sigma_i' = \phi_i$.
Each of the edges e_2, $\dots e_n$ occurs in two triangles in K. We glue the corresponding triangles together along the corresponding edges by

$$x_1 \sim x_2 \Leftrightarrow \phi(x_1) = \phi(x_2) \in e_i \text{ for some } i,\ 1 < i \leqslant n \text{ (or } x_1 = x_2).$$

Put $P = \bigsqcup \sigma_i'/{\sim}$. Then ϕ induces as usual a continuous map

$$\overline{\phi}\colon P \to |K|.$$

P is a topological polygon. The edges of triangles σ_i' which do not correspond to the e_i make up the edges of P. The map $\overline{\phi}$ identifies the edges of P in pairs. $\square$

2.15 Notation. We can represent a polygon with edges to be identified in pairs by a *symbol* as follows. Assign to each edge a letter, assigning the same letter to two edges if and only if they are to be identified. Now starting at any vertex write down the letters in sequence going round the boundary assigning the exponent -1 on the second appearance of any letter if and only if the order of the vertices is reversed. Conversely, given a symbol of this type in which each letter appears twice, there is a corresponding polygon with edges identified in pairs.

2.16 Example. We can write down the symbol corresponding to the polygon obtained in Example 2.13. The actual symbol depends on the choices made when constructing the polygon.

2.17 Remarks. (a) The topological space obtained by identifying the edges of a polygon in pairs depends up to homeomorphism only on the matched pairs of edges and the orders of the vertices. Thus the symbol determines the identification space up to homeomorphism.
(b) Identifying the edges of a polygon in pairs gives rise to a path-connected

closed surface. To prove that the quotient space is locally Euclidean we consider three types of points: (a) points arising from the points in the interior of the polygon (in this case the interior of the polygon provides the required open set); (b) points arising from identifying a pair of points in the interior of edges; (c) points arising from identifying several vertices.

(c) Different choices in constructing the topological polygon for Proposition 2.14 and in the notation of Notation 2.15 can lead to different symbols from the same simplicial surface.

2.18 Problem. When do two symbols represent homeomorphic topological surfaces?

2.19 Definition. Two symbols are *equivalent* if they represent the same surface up to homeomorphism.

2.20 Theorem. [**Classification Theorem for Surface Symbols**] Every symbol is equivalent to one and only one of

(i) xx^{-1},

(ii) $x_1y_1x_1^{-1}y_1^{-1}x_2y_2x_2^{-1}y_2^{-1}\ldots x_gy_gx_g^{-1}y_g^{-1}$ (where $g \geqslant 1$),

(iii) $x_1x_1x_2x_2\ldots x_gx_g$ (where $g \geqslant 1$).

Rules for the manipulation of symbols

2.21 Proposition. The following operations replace a symbol by an equivalent symbol:

(i) any letter can be renamed;

(ii) x can be replaced by x^{-1} (writing $(x^{-1})^{-1} = x$);

(iii) a symbol can be cyclically permuted;

(iv) a symbol can be formally inverted;

(v) $xx^{-1}U \sim U$;

(vi) $xUxV \sim xxU^{-1}V$;

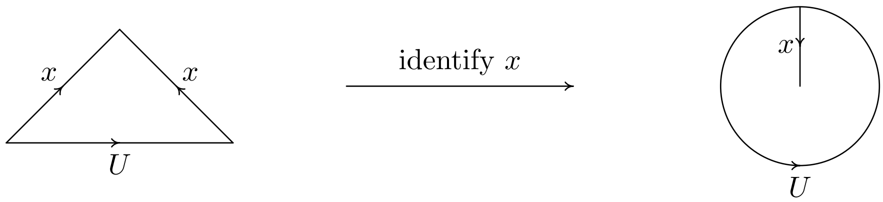
(vii) $xUVx^{-1}W \sim xVUx^{-1}W$.

Here U, V and W standard for sequences of letters (syllables) and U^{-1} is the formal inverse of U.

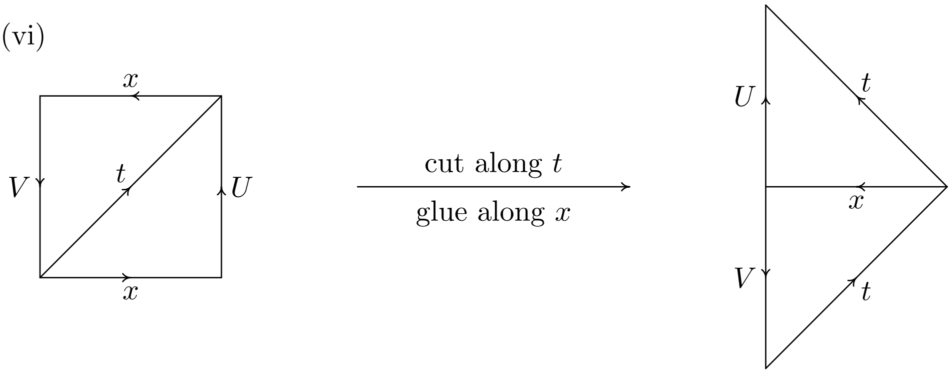
Proof. Parts (i), (ii), (iii) and (iv) follow immediately from different choices made when writing down the symbol from the polygon with edges to be identified in pairs (Notation 2.15).

The other parts are proved by cut and paste arguments indicated as follows.

(v)

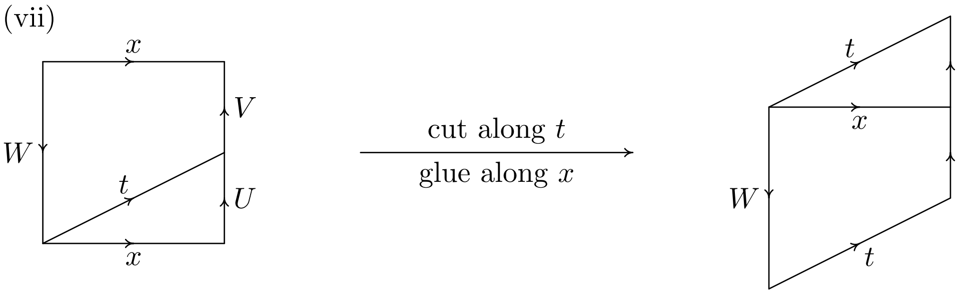


(vi)



Thus $xUxV \sim ttU^{-1}V \sim xxU^{-1}V$ (using (i)).

(vii)



Thus $xUVx^{-1}W \sim tVUt^{-1}W \sim xVUx^{-1}W$ (using (i)). $\square$

2.22 Corollary. Syllables of the form (i) xx, (ii) xUx^{-1} commute with every letter in a symbol (up to equivalence).

Proof. (i)

$$\begin{aligned}
UxxyV &\sim xxyVU && \text{(by 2.21(iii))} \\
&\sim xy^{-1}xVU && \text{(by 2.21(vi))} \\
&\sim U^{-1}V^{-1}x^{-1}yx^{-1} && \text{(by 2.21(iv))} \\
&\sim x^{-1}yx^{-1}U^{-1}V^{-1} && \text{(by 2.21(iii))} \\
&\sim x^{-1}x^{-1}y^{-1}U^{-1}V^{-1} && \text{(by 2.21(vi))} \\
&\sim VUyxx && \text{(by 2.21(iv))} \\
&\sim UyxxV && \text{(by 2.21(iii)).}
\end{aligned}$$

(ii)

$$\begin{aligned}
VxUx^{-1}yW &\sim x^{-1}yWVxU && \text{(by 2.21(iii))} \\
&\sim x^{-1}WVyxU && \text{(by 2.21(vii))} \\
&\sim VyxUx^{-1}W && \text{(by 2.21(iii))} .
\end{aligned}$$

□

2.23 Corollary. $x_1x_1x_2y_2x_2^{-1}y_2^{-1}U \sim x_1x_1x_2x_2x_3x_3U$.

Proof.

$$\begin{aligned}
x_1x_1x_2y_2x_2^{-1}y_2^{-1}U &\sim x_1x_2^{-1}x_1y_2x_2^{-1}y_2^{-1}U && \text{(by 2.21(vi))} \\
&\sim x_1x_2^{-1}x_2^{-1}y_2^{-1}x_1^{-1}y_2^{-1}U && \text{(by 2.21(vi))} \\
&\sim x_1x_2^{-1}x_2^{-1}y_2^{-1}y_2^{-1}x_1U && \text{(by 2.21(vi))} \\
&\sim x_1x_1x_2^{-1}x_2^{-1}y_2^{-1}y_2^{-1}U && \text{(by 2.22(i))} \\
&\sim x_1x_1x_2x_2x_3x_3U && \text{(by 2.21(i) and (ii).}
\end{aligned}$$

□

Proof of Theorem 2.20 (Reduction to standard form).
Step 1. Given a symbol W, by repeated application of 2.21(vi) and 2.22(i) we obtain an equivalent symbol NA where N is a symbol of type (iii) (or $\emptyset$) and A is a symbol in which each pair of equal letters has opposite exponents. If $A = \emptyset$ we have a symbol of the required form.
Step 2. We write $W \sim NM$ where N is as above and M is a symbol of type (ii) (or $\emptyset$). This is done by induction.
Suppose that $NA \sim NBC$ where B is of type (ii) (or $\emptyset$) and C has the property that equal letters have opposite exponents. If $C \neq \emptyset$ then $NBC = NBxDx^{-1}E$. If D contains a syllable of the form yUy^{-1} then this can be moved to the right of x^{-1} by 2.22(ii) and so we may assume that no two letters of D are the same.
If $D = \emptyset$ then we may cancel xx^{-1} by 2.21(v) [unless $N = B = E = \emptyset$ in which case the symbol is xx^{-1} of the required form (type (i)]. This reduces the length of C by 2.
If $D \neq \emptyset$ write $D = Fy$ so that

$$\begin{aligned}
NBC &= NBxFyx^{-1}Gy^{-1}H \\
&\sim NBxFyGx^{-1}y^{-1}H \text{ (by 2.21(vii))} \\
&\sim NBxGFyx^{-1}y^{-1}H \text{ (by 2.21(vii))} \\
&\sim NBxyx^{-1}y^{-1}GFH \text{ (by 2.22(ii))}
\end{aligned}$$

Thus in either case $NBC \sim NB'C'$ where B' is of type (ii) and C' is shorter than C.

Induction on the length of C shows that $W \sim NA \sim NM$, as required. If N or $M = \emptyset$ we have the required form.

Step 3. If N and M are both non-empty then NM is equivalent to a symbol of type (iii) by Corollary 2.23.

The proof that each symbol is equivalent to only one symbol in standard form will come later. □

2.24 Example. In the following example I have tried to indicate which letters are involved in applying each rule, by means of dots, brackets and braces.

$$\left.\begin{array}{ll}
a\dot{b}\underbrace{cdef}\dot{b}gdhe^{-1}gc^{-1}fa^{-1}h & \\
\sim a(bb)f^{-1}\dot{e}^{-1}\underbrace{d^{-1}c^{-1}gdh}\dot{e}^{-1}gc^{-1}fa^{-1}h & \text{by 2.21(vi)} \\
\sim a(bb)f^{-1}(e^{-1}e^{-1})h^{-1}d^{-1}g^{-1}cdgc^{-1}fa^{-1}h & \text{by 2.21(vi)} \\
\sim (bb)(e^{-1}e^{-1})\dot{a}\underbrace{f^{-1}h^{-1}d^{-1}g^{-1}cdgc^{-1}f}\dot{a}^{-1}h & \text{by 2.22(i)}
\end{array}\right\}\text{Step 1}$$

$$\left.\begin{array}{ll}
\sim (bb)(e^{-1}e^{-1})\underbrace{\dot{a}\dot{a}^{-1}}f^{-1}h^{-1}d^{-1}g^{-1}cdgc^{-1}fh & \text{by 2.22(ii)} \\
\sim (bb)(e^{-1}e^{-1})\dot{f}^{-1}h^{-1}\underbrace{d^{-1}g^{-1}cd}gc^{-1}\dot{f}h & \text{by 2.21(v)} \\
\sim (bb)(e^{-1}e^{-1})\dot{f}^{-1}h^{-1}g\dot{c}^{-1}\underbrace{\dot{f}(d^{-1}g^{-1})}\dot{c}dh & \text{by 2.22(ii)} \\
\sim (bb)(e^{-1}e^{-1})\dot{f}^{-1}(h^{-1}g\dot{c}^{-1})(d^{-1}g^{-1})\dot{f}\dot{c}dh & \text{by 2.21(vii)} \\
\sim (bb)(e^{-1}e^{-1})\dot{f}^{-1}d^{-1}g^{-1}h^{-1}g\underbrace{\dot{c}^{-1}\dot{f}\dot{c}}dh & \text{by 2.21(vii)} \\
\sim (bb)(e^{-1}e^{-1})(f^{-1}c^{-1}fc)\dot{d}^{-1}\underbrace{g^{-1}h^{-1}g}\dot{d}h & \text{by 2.22(ii)} \\
\sim (bb)(e^{-1}e^{-1})(f^{-1}c^{-1}fc)\underbrace{\dot{d}^{-1}\dot{d}}g^{-1}h^{-1}gh & \text{by 2.22(ii)} \\
\sim (bb)(e^{-1}e^{-1})(f^{-1}c^{-1}fc)(g^{-1}h^{-1}gh) & \text{by 2.21(v)}
\end{array}\right\}\text{Step 2}$$

$$\sim x_1x_1x_2x_2x_3x_3x_4x_4x_5x_5x_6x_6 \qquad \text{by 2.23 and 2.21(i)} \qquad \text{Step 3}$$

which is of the required standard form.

Geomtrical interpretation of the symbol classification theorem

2.25 Proposition. Suppose that W_1 and W_2 are symbols (with no letter in common) representing surfaces S_1 and S_2. Then the symbol W_1W_2 represents the connected sum $S_1\#S_2$.

Proof. The proof is by a cut and paste argument. □

2.26 Corollary. Every closed surface is homeomorphic to one of the standard surfaces listed in Theorem 1.15.

Proof. By the Triangulation Theorem (Theorem 2.7) every topological surface may be represented by a symbol and by the Symbol Classification Theorem this can be taken to be of the form of one of the symbols listed in that theorem. The result now follows from Proposition 2.25 once we observe that:

- the symbol xx^{-1} represents S^2 (by Problems 1, Question 5);
- the symbol $xyx^{-1}y^{-1}$ represents $T_1 = S^1 \times S^1$ (by Problems 1, Question 4);
- the symbol xx represents $P_1 = P^2$ (by the observation that $D^2/\sim \cong P^2$ in the proof of Proposition 1.20).

$\square$

2.27 Corollary. $P_1 \# T_1 \cong P_3$.

Proof. By Proposition 2.25 this is the geometrical version of Corollary 2.23. $\square$