Autumn Semester 2017–2018

MATH41071/MATH61071 *Algebraic topology*

§3. Topological Invariants of Surfaces

The Euler characteristic

3.1 Definition. Given a simplicial surface K with v vertices, e edges and f triangles, then the *Euler characteristic* of K, $\chi(K)$, is given by $\chi(K) = v - e + f$.

3.2 Theorem. If K_1 and K_2 are simplicial surfaces such that $|K_1| \cong |K_2|$, then $\chi(K_1) = \chi(K_2)$, i.e. the Euler characteristic of a closed surface is a topological invariant and for a closed surface S we can define $\chi(S) = \chi(K)$ where K is a simplicial surface such that $|K| \cong S$.

The proof of this will be discussed later.

3.3 Examples. (a) $\chi(S^2) = 4 - 6 + 4 = 2$ using the simplicial surface of Example 2.8(a).
(b) $\chi(T_1) = 9 - 27 + 18 = 0$ using the simplicial surface of Example 2.8(b)).
(c) $\chi(P_1) = 10 - 27 + 18 = 1$ using the simplicial surface of Example 2.8(d)).

3.4 Proposition. For closed surfaces S_1 and S_2, $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$.

3.5 Corollary. (i) $\chi(S^2) = 2$;
(ii) $\chi(T_g) = 2 - 2g$ for $g \geqslant 1$;
(iii) $\chi(P_g) = 2 - g$ for $g \geqslant 1$.

Proof of the Proposition 3.4. Suppose that K_1 and K_2 are simplicial surfaces such that $|K_1| \cong S_1$ and $|K_2| \cong S_2$. Then we may construct simplicial surface K such that $|K| \cong S_1 \# S_2$ by removing a triangle from each of K_1 and K_2 and identifying the vertices and edges of this triangle. Then:
$f(K) = f(K_1) + f(K_2) - 2$,
$e(K) = e(K_1) + e(K_2) - 3$,
$v(K) = v(K_1) + v(K_2) - 3$.
Hence $\chi(K) = \chi(K_1) + \chi(K_2) - 2 + 3 - 3 = \chi(K_1) + \chi(K_2) - 2$ as required.
$\square$

3.6 Remark. The Euler characteristic distinguishes the surfaces (or surface symbols) in list (ii) of Theorem 1.15 and Theorem 2.20 from each other (and from the surface in list (i)) and it distinguishes the surfaces in list (iii) from each other and from the surface in list (i)). However, if $\chi(S)$ is even and less than 2 then there are two possibilities for S, one in list (ii) and one in list (iii), for example the torus (T_1) and the Klein bottle (P_2) have the same Euler characteristic: $\chi(T_1) = \chi(P_2) = 0$.

Orientability

3.7 Definition. Given an triangle $\langle v_1, v_2, v_3 \rangle$ an *orientation* for the the triangle is given by a cyclic ordering of the vertices.
From now on we will work with oriented triangles and write $\sigma_1 = \sigma_2$ if σ_1 and σ_2 are the same triangle with the same orientation and $\sigma_1 = -\sigma_2$ if they are the same triangle with the opposite orientation. Thus,

$$\langle v_0, v_1, v_2 \rangle = \langle v_1, v_2, v_0 \rangle = \langle v_2, v_0, v_1 \rangle = -\langle v_2, v_1, v_0 \rangle = -\langle v_1, v_0, v_2 \rangle = -\langle v_0, v_2, v_1 \rangle.$$

Similarly, an *orientation* for an edge is given by an ordering of the vertices so that, working with oriented edges,

$$\langle v_0, v_1 \rangle = -\langle v_1, v_0 \rangle.$$

An orientation for a triangle induces an orientation for each of its edges. For example, the oriented triangle $\langle v_0, v_1, v_2 \rangle$ gives the orientations $\langle v_0, v_1 \rangle$, $\langle v_1, v_2 \rangle$ and $\langle v_2, v_0 \rangle$ on its edges. The opposite orientation for the triangle induces the opposite orientations on the edges.
Two triangles with a common edge are *coherently oriented* if the orientations induced on the common edge are opposite. For example, given the two oriented triangles $\langle v_1, v_2, v_3 \rangle$ and $\langle v_1, v_2, v_4 \rangle$ both have the same oriented common edge $\langle v_1, v_2 \rangle$ and so are *not* coherently oriented, whereas the the two oriented triangles $\langle v_1, v_2, v_3 \rangle$ and $\langle v_2, v_1, v_4 \rangle$ are coherently oriented since for the first the common edge is oriented $\langle v_1, v_2 \rangle$ whereas for the second triangle it is oriented $\langle v_2, v_1 \rangle = -\langle v_1, v_2 \rangle$.
An *orientation* of a simplicial surface is a choice of orientation for each triangle so that each pair of triangles with a common edge are coherently oriented. If a simplicial surface has an orientation it is *orientable*; otherwise it is *non-orientable*.
Notice that if a simplicial surface is orientable then an orientation is given by choosing an orientation for a single triangle; the orientation of all the other triangles will be determined by coherence across edges. So an orientable simplicial surface has two orientations.

3.8 Theorem. If K_1 and K_2 are simplicial surfaces such that $|K_1| \cong |K_2|$ then K_1 is orientable if and only if K_2 is orientable. Thus we can say that a closed surface S is orientable if and only if K is orientable where $|K| \cong S$ and orientability of a simplicial surface is a topological property.

The proof of this will be discussed later.

3.9 Examples. (a) An orientation of the simplicial surface of Example 2.8(a) is given by $\langle v_0, v_1, v_2\rangle$, $\langle v_1, v_0, v_3\rangle$, $\langle v_0, v_2, v_3\rangle$, $\langle v_2, v_1, v_3\rangle$. We simply have to check coherence across each of the six edges to confirm this: for example the edge with vertices v_2 and v_3 is oriented $\langle v_2, v_3\rangle$ by the third triangle and $\langle v_3, v_2\rangle$ by the fourth triangle. Hence S^2 is orientable.

(b) An orientation of the simplicial surface of Example 2.8(b) is given by orienting each triangle anticlockwise in the template. This ensures coherence across edges corresponding to internal edges of the template. We can then check coherence across edges corresponding the boundary edges of the template; for example, the edge between v_2 and v_2 is oriented $\langle v_3, v_2\rangle$ by the oriented triangle $\langle v_3, v_2, v_5\rangle$ and is oriented $\langle v_2, v_3\rangle$ by the oriented triangle $\langle v_2, v_3, v_9\rangle$. Hence K is orientable and so T_1 is orientable.

(c) If we orient the top left hand triangle of the simplicial surface of Example 2.8(d) as $\langle v_1, v_5, v_2\rangle$ then, if this is part of an orientation of the the simplicial surface, coherence across edges corresponding to internal edges of the template forces all of the triangles to be oriented anticlockwise. However, this does not give an orientation since is is not coherent across all the edges corresponding to the boundary edges of the template (in fact, in this case, it is not coherent across any of these edges). For example the edge with vertices v_2 and v_3 is oriented $\langle v_3, v_2\rangle$ by the oriented triangle $\langle v_3, v_2, v_6\rangle$ and also by the oriented triangle $\langle v_3, v_2, v_{10}\rangle$ and so these two triangles are not coherently oriented. Precisely the same argument works if we start by orienting the top left hand triangle as $\langle v_1, v_2, v_5\rangle$ since then internal coherence forces all of the triangles to be oriented clockwise. Hence K is non-orientable and so P_1 is non-orientable.

3.10 Proposition. Given closed surfaces S_1 and S_2, $S_1 \# S_2$ is orientable if and only if S_1 and S_2 are orientable.

3.11 Corollary. (i) S^2 is orientable;

(ii) T_g is orientable for $g \geqslant 1$;

(iii) P_g is non-orientable for $g \geqslant 1$.

Proof of Proposition 3.10. $\Rightarrow$: Suppose that S_1 and S_2 are orientable. Suppose that $|K_1| \cong S_1$ and $|K_2| \cong S_2$. Then, as in the proof of Proposition 3.4, we may obtain K such that $|K| \cong S_1 \# S_2$ by removing a triangle $\langle v_1, v_2, v_3\rangle$ from K_1 and a triangle $\langle v_1', v_2', v_3'\rangle$ from K_2 and identifying the vertices $v_i \sim v_i'$ and the edges between them. Chose an orientation for

K_1 choosing the oriented triangle $\langle v_1, v_2, v_3\rangle$ and extending over K_1 by coherence. Similarly for K_2 starting with the oriented triangle $\langle v_3', v_2', v_1'\rangle$. The resulting orientation of the triangles of K is coherent since for example the triangle of $K_1 \setminus \{\langle v_1, v_2, v_3\rangle\}$ which has an edge with vertices v_1 and v_2 induces the orientation $\langle v_2, v_1\rangle$ on this edge whereas the triangle of $K_2 \setminus \{\langle v_1', v_2', v_3'\rangle\}$ which has an edge with vertices v_1' and v_2' induces the orientation $\langle v_1', v_2'\rangle$ on this edge giving coherence across the corresponding edge of K.

$\Leftarrow$: For the converse, suppose that $S_1 \# S_2$ is orientable. So using the notation of the previous part of the proof K is orientable so suppose that we have chosen an orientation. Then this orients the triangles of $K_1 \setminus \{\langle v_1, v_2, v_3\rangle\}$ and $K_2 \setminus \{\langle v_1', v_2', v_3'\rangle\}$. To complete the proof we observe that if all but one of the triangles of a triangle are oriented coherently then that last triangle can be oriented to complete the orientation, i.e. the induced orientations on the edges of this triangle must by cyclic. This is forced by the link condition at each vertex of the triangle. [Exercise] $\square$

3.12 Remarks. (a) Given the topological invariance of the Euler characteristic and orientability (Theorems 3.2 and 3.8), this completes the proof of Theorem 1.15 and Theorem 2.20 since orientability distinguishes the surfaces (or surface symbols) in lists (i) and (ii) from those in list (iii).

(b) If we are given a simplicial surface K it is usually easier the topological type of the underlying space $|K|$ by determining $\chi(K)$ and (if $\chi(K)$ is even) determining whether or not K is orientable. Notice that if $\chi(K)$ is odd then the surface is necessarily non-orientable and the value of the Euler characteristic determines the topological type of the surface.

(c) We say that T_g is an *orientable surface of genus* g and P_g is a *non-orientable surface of genus* g. To fit in with this we say that S^2 is an orientable surface of genus 0.

(d) It can be shown that a closed surface S is non-orientable if and only if it has a subspace homeomorphic to the Möbius band.

Stellar subdivision, the Hauptvermutung and the invariance of the Euler characteristic and orientability

3.13 Definition/Proposition. Let K be a simplicial surface.

(i) Suppose that v is an interior point of a triangle $\langle v_1, v_2, v_3\rangle$. A new simplicial surface K' may be obtained by

$$K' = K \setminus \{\langle v_1, v_2, v_3\rangle\} \cup \{\langle v_1, v_2, v\rangle,\ \langle v_1, v_3, v\rangle,\ \langle v_2, v_3, v\rangle\}.$$

We say that K' is obtained from K by *starring at* v.

(ii) Suppose that v is an interior point of an edge $\langle v_1, v_2\rangle$ which is in the triangles $\langle v_1, v_2, v_3\rangle$ and $\langle v_1, v_2, v_4\rangle$ then the simplicial surface K' obtained from K by *starring at* v is given by

$$K' = K \backslash \{\langle v_1, v_2, v_3\rangle,\ \langle v_1, v_2, v_4\rangle\} \cup \{\langle v_1, v_3, v\rangle,\ \langle v_2, v_3, v\rangle,\ \langle v_1, v_4, v\rangle,\ \langle v_2, v_4, v\rangle\}.$$

(iii) We say that K' is obtained from K *stellar subdivision* if is obtained by successively starring.

Proof. We have to show, that starring gives indeed a new simplicial surfaces. To see this, observe, that for the new vertex the link is formed by the edges of the removed triangle in case (i) and by the four remaining edges of the two removed triangles in case (ii). In both cases they form a simple closed polygon.

Consider the link in K' of one of the old vertices. It does only change when the vertex is contained in one of the removed triangles, but in this case the opposite edge in these removed triangles is just replaced by two of the new edges. Hence, the link condition for an old vertex is fulfilled after the starring if and only if it was fulfilled before. □

3.14 Proposition. If K' is obtained from K by stellar subdivision then

(a) $|K| = |K'|$;

(b) $\chi(K) = \chi(K')$;

(c) K is orientable if and only if K' is orientable.

Proof. It is only necessary to confirm that these results hold when we star once.

(a) This is obvious from the construction.

(b) If we star at an interior point of a triangle then $v' = v + 1$, $e' = e + 3$, $f' = f + 2$ and so $v' - e' + f' = v - e + f$.

If we star at an interior point of an edge then $v' = v+1$, $e' = e+3$, $f' = f+2$ and so the same result holds.

(c) Suppose we star at an interior point v of the oriented triangle $\langle v_1, v_2, v_3\rangle$. Then if the new triangles are oriented $\langle v_1, v_2, v\rangle$, $\langle v_1, v, v_3\rangle$, $\langle v, v_2, v_3\rangle$ then they are mutually coherent and induce the same orientations on the edges of $\langle v_1, v_2, v_3\rangle$. So starring at v cannot affect the orientability of the simplicial surface.

There is a similar argument for starring at an interior point of an edge. [Exercise] □

3.15 Theorem [The Hauptvermutung for surfaces, Poincaré 1904]. Given two simplicial surfaces K_1 and K_2 such that $|K_1| \cong |K_2|$ then there exist simplicial surfaces K_1' and K_2', obtained from K_1 and K_2 by stellar subdivision which are isomorphic in the sense of Remark 2.9.

Proof, Omitted. □

3.16 Remark. (a) This is a difficult result to prove. The dimension 3 analogue was proved in 1948 but in 1961 Milnor proved that $S^3 \times S^1 \times S^1$ has two triangulations without a common stellar subdivision and so the result is not true for manifolds of dimension 5 (and, in fact, for dimensions above 5).

(b) This course will describe an alternative approach to the invariance of the Euler characteristic and the orientability via homology theory, an approach which can be extended to all dimensions. This will take up the rest of the course.

3.17 Corollary. If K_1 and K_2 are simplicial surfaces such that $|K_1| \cong |K_2|$ then $\chi(K_1) = \chi(K_2)$ and K_1 is orientable if and only if K_2 is orientable.

Proof. This is immediate from the Theorem since $\chi(K_1) = \chi(K_1') = \chi(K_2') = \chi(K_2)$ and K_1 is orientable if and only if K_1' is orientable if and only if K_2' is orientable if and only if K_2 is orientable. $\square$