Autumn Semester 2017–2018

MATH41071/MATH61071 *Algebraic topology*

§T. Background material: Topology

For convenience this is an overview of basic topological ideas which will be used in the course. This material was covered in MATH31052 *Topology* and more details can be found in the notes for that course.

Basic definitions

T.1 Definition. A set τ of subsets of a set X is a *topology* on X if it has the following properties:

(i) the intersection of any finite set of elements of τ is in τ;

(ii) the union of any set of elements of τ is in τ;

(iii) $\emptyset \in \tau$ and $X \in \tau$.

A pair (X, τ) consisting of a set X and a topology τ on X is called a *topological space*. We usually refer to 'the space X' when the topology is clear.

The elements of τ are called the *open sets* of the topology.

T.2 Definition. Suppose that (X, τ_1) and (Y, τ_2) are topological spaces. Then a function $f\colon X \to Y$ is *continuous* (with respect to the topologies τ_1 and τ_2) if

$$U \in \tau_2 \Rightarrow f^{-1}(U) \in \tau_1,$$

i.e. the inverse image of each open set in Y is an open set in X.

T.3 Example. We may define a topology on Euclidean n-space $\mathbb{R}^n$ as follows. A subset $U \subset \mathbb{R}^n$ is open if and only if, for each $\mathbf{x}_0 \in U$, there is a positive $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}_0) = \{\, \mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x} - \mathbf{x}_0| < \varepsilon \,\} \subset U$. This is called the *usual topology* on $\mathbb{R}^n$.

T.4 Definition. A *basis* for a topology τ is a subset $\mathcal{B}$ of τ (i.e. a set of open sets in the topology) such that each open set (element of τ) can be expressed as a union of a collection of sets in $\mathcal{B}$.

A topological space which has a countable basis is said to be *second countable*.

T.5 Example. In $\mathbb{R}^n$ the set of all open balls with rational radii and centres with rational coordinates is a basis for the usual (metric) topology on $\mathbb{R}^n$. Hence $\mathbb{R}^n$ with the usual topology is second countable.

This makes use of the fact that the set of rationals $\mathbb{Q}$ is a countable set and the product of two countable sets is countable.

T.6 Definition. A subset A in a topological space X is *closed* if and only if its complement $X \setminus A$ is open.

T.7 Proposition. A function $f\colon X \to Y$ between topological spaces is continuous if and only if the inverse image of each closed set in Y is a closed set in X.

T.8 Definition. Suppose that X and Y are topological spaces. A function $f\colon X \to Y$ is a *homeomorphism* (or *topological equivalence*) if it is a continuous bijection with a continuous inverse. If such a homeomorphism exists then we say that X and Y are *homeomorphic* and write $X \cong Y$.

T.9 Proposition. A bijection $f\colon X \to Y$ of topological spaces is a homeomorphism if and only if either of the following conditions hold:

(a) U is open in X if and only if $f(U)$ is open in Y;

(b) A is closed in X if and only if $f(A)$ is closed in Y.

A sufficient condition for a continuous bijection to be a homeomorphism

T.10 Definition. A topological space is *Hausdorff* if, for each pair of distinct points x_1 and x_2 of X there are open sets U_1 and U_2 of X such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

T.11 Proposition. Euclidean space $\mathbb{R}^n$ with the usual topology is Hausdorff.

T.12 Definition. A collection $\mathcal{A}$ is subsets of X is a *covering* for $A \subset X$ if $\bigcup_{U \in \mathcal{A}} U \supset A$. If $\mathcal{A}$ and $\mathcal{B}$ are coverings for A and $\mathcal{B} \subset \mathcal{A}$ then $\mathcal{B}$ is a *subcovering* of $\mathcal{A}$. A covering $\mathcal{A}$ is *finite* if the number of subsets in $\mathcal{A}$ is finite.

A subset A of a topological space X is *compact* if each covering of A by open sets of X has a finite subcovering. Notice that X itself is a subset of X and if it is compact we refer to X as a *compact space*.

T.13 Theorem [Heine-Borel-Lebesgue]. A subset A in $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

T.14 Proposition. A closed subset of a compact space is compact.

T.15 Proposition. A compact subset of a Hausdorff space is closed.

T.16 Proposition. Suppose that $f\colon X \to Y$ is continuous and A is a compact subset of X. Then $f(A)$ is a compact subset of Y.

T.17 Theorem. Suppose that $f\colon X \to Y$ is a continuous bijection from a compact space X to a Hausdorff space Y. Then f is a homeomorphism.

Proof. This is an easy corollary of the previous three results. To prove that f is a homeomorphism, it is sufficient by Proposition T.9 to prove that if A is closed in X then $f(A)$ is closed in Y since f is a continuous bijection. However, since A is closed in X and X is compact it follows from Proposition T.14 that A is compact. Therefore, by Proposition T.16, $f(A)$ is compact. It then follows, by Proposition T.15, that $f(A)$ is closed in Y as required, since Y is Hausdorff. $\square$

Constructing new topological spaces

Subspaces

T.18 Definition. Suppose that (X, τ) is a topological space and X_1 is a subset of X. Then the *subspace topology* on X_1 is given by $V \subset X_1$ is open in X_1 if and only if $V = U \cap X_1$ for some open set U in X. We call X_1 with the subspace topology a *subspace* of X.

T.19 Proposition [Universal property of the subspace topology]. Given a topology space X with a subspace $X_1 \subset X$, the inclusion map $i\colon X_1 \to X$ is continuous and, given any map $f\colon Y \to X_1$ from a topological space Y, f is continuous if and only if the composition $i \circ f\colon Y \to X$ is continuous.

T.20 Proposition [Gluing lemma]. Suppose that X_1 and X_2 are subspaces of a topological space X such that $X = X_1 \cup X_2$ and both are closed in X [or both are open in X]. Given continuous maps $f_1\colon X_1 \to Y$, $f_2\colon X_2 \to Y$ such that $f_1(x) = f_2(x)$ for $x \in X_1 \cap X_2$. Then the function $f\colon X \to Y$ defined by $f(x) = f_i(x)$ for $x \in X_i$ is well-defined and continuous.

T.21 Proposition. A subspace of a Hausdorff space is Hausdorff.

T.22 Proposition. A subspace of a second countable topological space is second countable.

T.23 Proposition. A subspace X_1 of a topological space X is a compact space if and only if X_1 is a compact subset of X.

Product spaces

T.24 Definition. Suppose that X_1 and X_2 are topological spaces. The *product topology* on the cartesian product $X = X_1 \times X_2$ is the topology with a basis consisting of all sets of the form $U_1 \times U_2$ where U_i is open in X_i. We call $X_1 \times X_2$ with the product topology the *product* of the spaces X_1 and X_2.

T.25 Remark. The product topology on $\mathbb{R}^n$ given by the usual topology on $\mathbb{R}$ (defined by induction on n) is the same as the usual topology.

T.26 Proposition [Universal property of the product topology]. Suppose that $X = X_1 \times X_2$ is the product of two topological spaces X_1 and X_2. Then the two projection maps $p_i \colon X \to X_i$ ($(x_1, x_2) \mapsto x_i$) are continuous and, given any map $f \colon Y \to X$ from a topological space Y, f is continuous if and only if the two maps $p_i \circ f \colon Y \to X_i$ are continuous.

T.27 Proposition. Given two Hausdorff spaces X_1 and X_2, the product space $X_1 \times X_2$ is Hausdorff.

T.28 Proposition. Given two second countable spaces X_1 and X_2, the product space $X_1 \times X_2$ is second countable.

T.29 Theorem. Given two compact spaces X_1 and X_2, the product space $X_1 \times X_2$ is compact.

Disjoint unions

T.30 Definition. Given topological spaces X_1 and X_2 such that $X_1 \cap X_2 = \emptyset$ we write $X_1 \sqcup X_2 = X_1 \cup X_2$ to include the information that the sets are disjoint. The *disjoint union* topology on $X_1 \sqcup X_2$ is given by $U \subset X_1 \sqcup X_2$ is open if and only if $U \cap X_i$ is open in X_i for $i = 1, 2$. We call $X_1 \sqcup X_2$ with the disjoint union topology the *disjoint union* of X_1 and X_2.

T.31 Proposition [Universal property of the disjoint union topology]. Suppose that $X_1 \cup X_2$ is the disjoint union of topological spaces X_1 and X_2. Then a function $f\colon X_1 \sqcup X_2 \to Y$ to a topological space Y is continuous if and only if the restrictions $f|X_i\colon X_i \to Y$ are continuous.

T.32 Remark. We often abuse notation and write $X_1 \sqcup X_2$ when the spaces X_1 and X_2 are not disjoint. In this case we replace the spaces by homeomorphic copies which are disjoint. So for example we might write $S^n \sqcup S^n$ to mean something which we should write $(S^n \times \{1\}) \sqcup (S^n \times \{2\})$.

T.33 Proposition. Given two Hausdorff spaces X_1 and X_2, the disjoint union $X_1 \sqcup X_2$ is Hausdorff.

T.34 Proposition. Given two second countable spaces X_1 and X_2, the disjoint union $X_1 \sqcup X_2$ is second countable.

T.35 Proposition. Given two compact spaces X_1 and X_2, the disjoint union $X_1 \sqcup X_2$ is compact.

Identification spaces

T.36 Definition. Suppose that $q\colon X \to Y$ is a surjection from a topological space X to a set Y. Then the *quotient topology* on Y determined by X is given by $V \subset Y$ is open if and only if $q^{-1}(V)$ is open in X. We call Y with the quotient topology a *quotient space* of X.

T.37 Proposition [Universal property of the quotient topology]. Suppose that $q\colon X \to Y$ is a surjection of topological spaces and that Y has the quotient topology determined by q. Then q is continuous and a function $f\colon Y \to Z$ to a topological space Z is continuous if and only if $f \circ q\colon X \to Z$ is continuous.

T.38 Definition. An *equivalence relation* on a set X is a relation $\sim$ which satisfies the following properties for all x, y, $z \in X$,

reflexivity $x \sim x$;

symmetry if $x \sim y$ then $y \sim x$;

transitivity if $x \sim y$ and $y \sim z$ then $x \sim z$.

T.39 Proposition. An equivalence relation $\sim$ on a set X generates a partition of the set X into disjoint equivalence classes. For $x_0 \in X$, the equivalence class of x_0, $[x_0]$, is defined by $[x_0] = \{ x \in X \mid x \sim x_0 \}$. We write $X/\sim$ for the set of equivalence classes.

T.40 Definition. Suppose that $\sim$ is an equivalence relation on a topological space X. The *quotient topology* on the set of equivalence classes $X/\sim$ is the quotient topology determined by the function $q\colon X \to X/\sim$, $x \mapsto [x]$. With this topology we call $X/\sim$ an *identification space*.

T.41 Example. Define an equivalence relation on $I^2 = [0,1]^2$ by $(s,t) \sim (s,t)$ for all $(s,t) \in I^2$, $(s,0) \sim (s,1)$, $(s,1) \sim (s,0)$, $(0,t) \sim (1,t)$ and $(1,t) \sim (0,t)$. [We will normally describe this as the equivalence relation generated by $(s,0) \sim (s,1)$ and $(0,t) \sim (1,t)$ since the other relations are then automatic.] Then the identification space $I^2/\sim$ is homeomorphic to the product space $S^1 \times S^1$ known as the *torus*.

A homeomorphism $F\colon I^2/\sim \to S^1 \times S^1$ is induced by the continuous map $f\colon I^2 \to S^1 \times S^1$ given by $f(s,t) = \big(\exp(2\pi i s), \exp(2\pi i t)\big)$ (continuous by the universal property of the product topology since the exponential function is continuous).

This map given by $F([s,t]) = f(s,t)$ is a well-defined bijection since f is a surjection such that $f(s_1,t_1) = f(s_2,t_2) \Leftrightarrow (s_1,t_1) \sim (s_2,t_2)$. It is continuous since f is continuous by the universal property of the quotient topology (since $F \circ q = f$). It is then a homeomorphism by Theorem T.17 since $S^1 \times S^1$ is Hausdorff (a product of Hausdorff spaces by Proposition T.11 and Proposition T.21) and $I^2/\sim$ is compact (since I^2 is compact by Theorem T.13 and the Proposition T.42 below which is a simple corollary of Proposition T.16).

T.42 Proposition. If X is a compact space, then any identification space $X/\sim$ is also compact.

T.43 Remark. The Hausdorff and second countable properties are not necessarily inherited by quotient spaces.