Autumn Semester 2017–2018

MATH41071/MATH61071 *Algebraic topology*

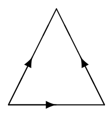
Problems 4: Simplicial complexes

1. Prove that if $x \in \langle v_0, v_1, \ldots, v_r\rangle$ is a point in an r-simplex then x can be written uniequely in the form $x = \sum_{i=0}^{r} t_i v_i$ where $\sum_{i=0}^{r} t_i = 1$. [Proposition 4.9]

2. Prove that an isomorphism $f\colon K_1 \to K_2$ of geometric simplicial complexes induces a homeomorphism $|f|\colon |K_1| \to |K_2|$ of their underlying spaces. [Corollary 4.11]

3.(*) Describe triangulations of the closed cylinder $I^2/(s,0) \sim (s,1)$ and of the Möbius band $I^2/(s,0) \sim (1-s,1)$.

4. The *dunce hat* is obtained by identifying all three sides of a triangle as shown. Construct a simplicial complex which triangulates this space.



5.(*) Given a simplicial complex K with n_r r-simplices for $0 \leqslant r \leqslant \dim K$, the *Euler characteristic* $\chi(K)$ of K is defined by

$$\chi(K) = \sum_{r=0}^{\dim K} (-1)^r n_r.$$

Let $s_n = \langle v_0, v_1, \ldots, v_n\rangle$, an n-simplex. Find the Euler characteristic of the following simplicial complexes:

(i) $\bar{\Delta}^n$ (for $n \geqslant 0$);

(ii) $(\bar{\Delta}^{[n-1]})$ (the $(n-1)$-skeleton of $\bar{\Delta}^n$, for $n \geqslant 1$);

(iii) $(\bar{\Delta}^{[n-2]})$ (the $(n-2)$-skeleton of $\bar{\Delta}^n$, for $n \geqslant 2$);

(iv) the simplicial complex you constructed in Question 4.

6. Prove that the standard n-simplex Δ^n is homeomorphic to the n-ball D^n.

[Hint: Produce a sequence of homeomorphisms

$$\Delta^n \cong \left\{(t_1,\dots,t_n)\in\mathbb{R}^n \;\middle|\; t_i \geqslant 0,\ \sum_{i=1}^n t_i \leqslant 1\right\} \cong I^n = \{\,(t_1,\dots,t_n) \mid t_i \geqslant 0, \max(t_i) \leqslant 1\,\}$$

$$\cong [-1,1]^n = \{\,(t_1,\dots,t_n) \mid t_i \geqslant 0, \max(|t_i|) \leqslant 1\,\} \cong D^n = \left\{\,(t_1,\dots,t_n) \;\middle|\; \sum_{i=1}^n t_i^2 \leqslant 1\,\right\}$$

or you may find something simpler.]