Autumn Semester 2017–2018

MATH41071/MATH61071 *Algebraic topology*

Problems 6: Homotopy and simplicial approximation

1. Using your simplicial complex K from Problems 5, Question 3(v), find the subcomplex L such that $|L|$ gives the boundary circle of the Möbius band. Find the map of simplicial homology groups $H_1(L) \to H_1(K)$ induced by the inclusion map $L \to K$.

2. Prove that *homotopy* is an equivalence relation on the set of continuous functions from a topological space X to a topological space Y.

3. Prove that if a topological space X is contractible then

(i) X is path-connected;

(ii) for every point $x_0 \in X$, $\{x_0\}$ is a deformation retract of X.

4.

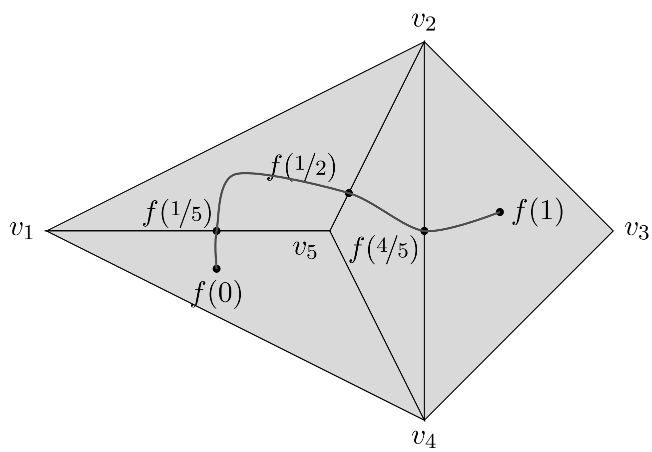
(a) Let K be the geometric simplicial complex in $\mathbb{R}$ with vertices 0, 1/3 and 1 and edges $\langle 1, 1/3\rangle$ and $\langle 1/3, 1\rangle$, and L be the geometric simplicial complex in $\mathbb{R}$ with vertices 0, 2/3 and 1 and edges $\langle 1, 2/3\rangle$ and $\langle 2/3, 1\rangle$. Thus $|K| = |L| = [0,1]$. Let $f\colon [0,1] \to [0,1]$ be the function $f(x) = x^2$. Prove that

(i) $f\colon |K| \to |L|$ does not have a simplicial approximation

(ii) $f\colon |K'| \to |L|$ does not have a simplicial approximation.

(iii) $f\colon |K''| \to |L|$ does have a simplicial approximation.

(b) Consider the simplicial complex L vertices v_1, v_2, v_3, v_4, v_5, which is sketched below, and an injective continuous map $f : [0,1] \to |L|$ having the image being indicated in the picture. Let K be the simplicial complex consisting just of the simplex $\langle 0, 1\rangle$ and its faces. Give a simplicial approximation for f on a sufficiently fine barycentric subdivision $K^{(m)}$ of K.



6. Use the simplicial approximation theorem to prove that all continuous functions $S^m \to S^n$ are homotopic to a constant function and so that there is only one homotopy class of functions $S^m \to S^n$ if $0 \leqslant m < n$.