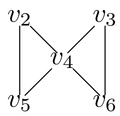
Autumn Semester 2017–2018

MATH41071/MATH61071 *Algebraic topology*

Solutions 2

1. This satisfies the intersection condition (automatically) and the connectivity condition (a path through all the vertices is given by going along edges v_4 to v_3 to v_1 to v_2 to v_5 to v_6). The link condition fails for all of the vertices. For example the link of v_1 is given by the six edges $\langle v_2, v_4\rangle$, $\langle v_2, v_5\rangle$, $\langle v_3, v_4\rangle$, $\langle v_3, v_6\rangle$, $\langle v_4, v_5\rangle$ and $\langle v_4, v_6\rangle$ which form two triangles meeting at v_4 which is not a simple closed polygon.



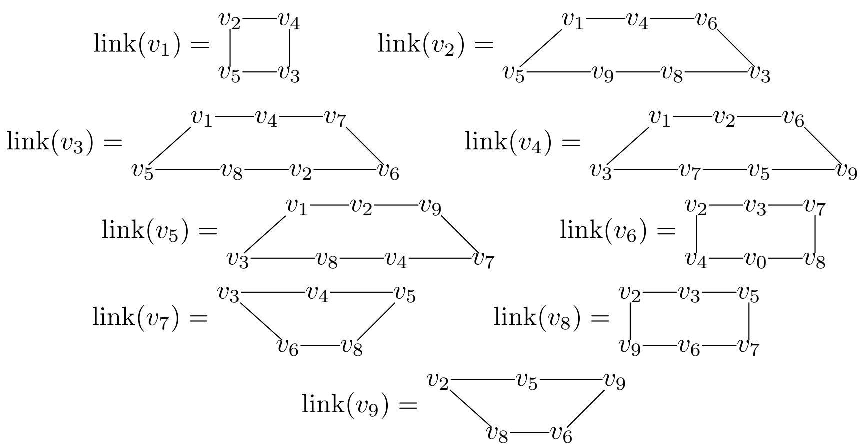
2. Suppose that $\langle v, v'\rangle$ is an edge in a simplicial surface. Note that if this edge lies in a triangle $\langle v, v', w\rangle$ then $\langle v', w\rangle$ is in the link of v. The link of v is a simple closed polygon consisting of edges $\langle v_1, v_2\rangle$, $\langle v_2, v_3\rangle$, $\ldots \langle v_{n-1}, v_n\rangle$, $\langle v_n, v_1\rangle$. The vertex v' must be one of the vertices v_i, $1 \leqslant i \leqslant n$, say v_k. Then the edge $\langle v, v'\rangle$ lies just in the triangles $\langle v, w, v_{k-1}\rangle$ and $\langle v, w, v_{k+1}\rangle$ (indexing the vertices of the link modulo n).

3. The intersection condition is satisfied automatically for all of these sets of triangles by the position of the vertices.

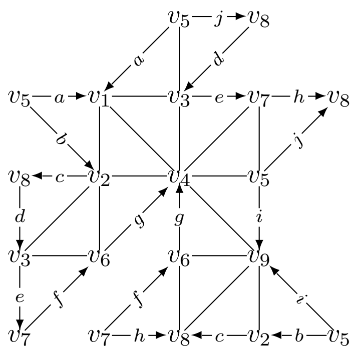
(a) The connectivity condition is satisfied since a path through all the vertices is given by the following.

$$v_5 \text{---} v_1 \text{---} v_2 \text{---} v_4 \text{---} v_3 \text{---} v_6 \text{---} v_7 \text{---} v_8 \text{---} v_9$$

We can check the link condition for each of the vertices.



Hence K is a simplicial surface. We can now represent $|K|$ as a polygon with edges identified in pairs as follows (there are lots of possibilities here).



Reducing the symbol for this polygon to standard form gives the following.

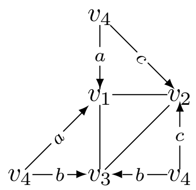
$$
\begin{aligned}
& (aa^{-1})bcde(fgg^{-1}f^{-1})hc^{-1}b^{-1}(ii^{-1})jh^{-1}e^{-1}d^{-1}j^{-1} \\
\sim\ & \dot{b}(cdehc^{-1})\dot{b}^{-1}jh^{-1}e^{-1}d^{-1}j^{-1} \text{ (by 2.21(v))} \\
\sim\ & (bb^{-1})\dot{c}de\dot{h}(\dot{c}^{-1})(j)\dot{h}^{-1}e^{-1}d^{-1}j^{-1} \text{ (by 2.22(ii))} \\
\sim\ & \dot{c}(de\dot{h})(j)\dot{c}^{-1}\dot{h}^{-1}e^{-1}d^{-1}j^{-1} \text{ (by 2.21(v),(vii))} \\
\sim\ & \dot{c}jde(\dot{h}\dot{c}^{-1}\dot{h}^{-1})e^{-1}d^{-1}j^{-1} \text{ (by 2.21(vii))} \\
\sim\ & (chc^{-1}h^{-1})(jdee^{-1}d^{-1}j^{-1}) \text{ (by 2.22(ii))} \\
\sim\ & chc^{-1}h^{-1} \text{ (by 2.21(v))} \\
\sim\ & xyx^{-1}y^{-1} \text{ (by 2.21(i))}
\end{aligned}
$$

Hence the surface is homeomorphic to the torus.

(b) The connectivity condition is satisfied since a path through all the vertices is given as follows.

$$v_1 \text{——} v_2 \text{——} v_3 \text{——} v_4$$

Each link is a triangle and so the link condition is satisfied. Hence K is a simplicial surface.
We can now represent $|K|$ as a polygon with edges identified in pairs as follows.



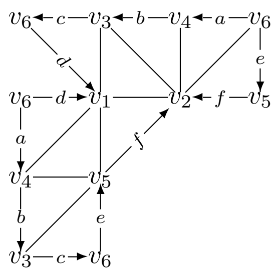
Reducing a symbol to standard form gives $aa^{-1}bb^{-1}cc^{-1} \sim aa^{-1}$ and so the surface is homeomorphic to the sphere.

(c) The connectivity condition is satisfied since a path through all the vertices is as follows.

$$v_4 \text{——} v_5 \text{——} v_1 \text{——} v_2 \text{——} v_3 \text{——} v_6$$

We can check the link condition for each of the vertices and find that each is a pentagon. Hence K is a simplicial surface.

We can now represent $|K|$ as a polygon with edges identified in pairs as follows.

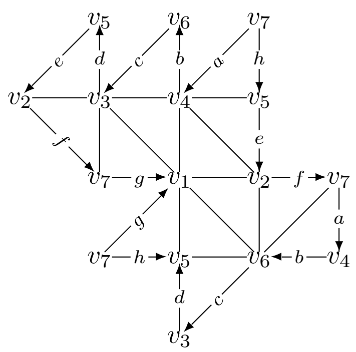


Reducing a symbol to standard form gives $abcdd^{-1}abceff^{-1}e^{-1} \sim \dot{a}bc\dot{a}bc \sim aac^{-1}b^{-1}bc \sim aa$ and so the surface is homeomorphic to the projective plane.

(d) The connectivity condition is satisfied since a path through all the vertices is as follows.

$$v_5 \text{———} v_6 \text{———} v_1 \text{———} v_2 \text{———} v_4 \text{———} v_3 \text{———} v_7$$

We can check the condition for each of the vertices and find that each is a hexagon. Hence K is a simplicial surface.
We can now represent $|K|$ as a polygon with edges identified in pairs as follows.

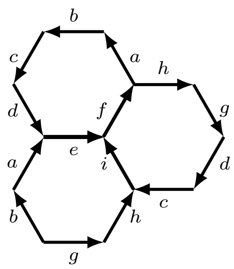


Reducing a symbol to standard form gives

$$\begin{aligned}
abcdef(gg^{-1})hd^{-1}c^{-1}b^{-1}a^{-1}f^{-1}e^{-1}h^{-1} &\sim \dot{a}(bcdefhd^{-1}c^{-1}b^{-1})\dot{a}^{-1}f^{-1}e^{-1}h^{-1}\\
&\sim (aa^{-1})\dot{b}(cdefhd^{-1}c^{-1})\dot{b}^{-1}f^{-1}e^{-1}h^{-1}\\
&\sim (bb^{-1})\dot{c}(defhd^{-1})\dot{c}^{-1}f^{-1}e^{-1}h^{-1}\\
&\sim (cc^{-1})\dot{d}ef\dot{h}(\dot{d}^{-1})(f^{-1}e^{-1})\dot{h}^{-1}\\
&\sim \dot{d}(ef\dot{h})(f^{-1}e^{-1})\dot{d}^{-1}\dot{h}^{-1}\\
&\sim \dot{d}f^{-1}e^{-1}ef(\dot{h}\dot{d}^{-1}\dot{h}^{-1})\\
&\sim (dhd^{-1}h^{-1})f^{-1}e^{-1}ef\\
&\sim xyx^{-1}y^{-1}
\end{aligned}$$

and so the surface is homeomorphic to the torus.

4. We can produce a single polygon with edges to be identified in pairs by using three edge identifications to join up the polygons as follows.

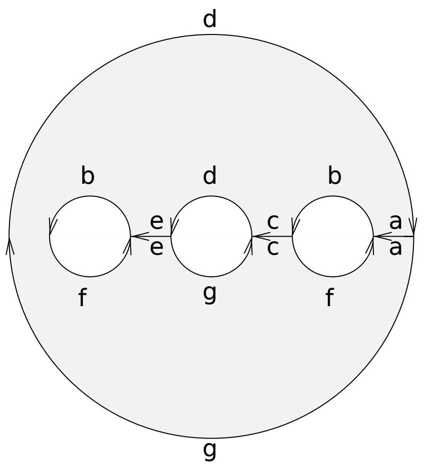


Now reducing the symbol for the polygon identifications to standard form gives the following.

$$
\begin{aligned}
abcd(a^{-1})(b^{-1}ghc^{-1})d^{-1}g^{-1}c^{-1} &\sim a(bcd)(b^{-1}ghc^{-1})a^{-1}d^{-1}g^{-1}h^{-1} \\
&\sim ab^{-1}ghc^{-1}bc(da^{-1}d^{-1})g^{-1}h^{-1} \\
&\sim (ada^{-1}d^{-1})b^{-1}gh(c^{-1}bc)g^{-1}h^{-1} \\
&\sim (ada^{-1}d^{-1})(b^{-1}c^{-1}bc)(ghg^{-1}h^{-1}) \\
&\sim (x_1y_1x_1^{-1}y_1^{-1})(x_2y_2x_2^{-1}y_2^{-1})(x_3y_3x_3^{-1}y_3^{-1}).
\end{aligned}
$$

Hence the surface is orientable of genus 3.

5. The disc with three smaller discs removed may be cut as shown to form a polygon with edges to be identified in pairs.



Now reducing the symbol for the polygon identifications to standard form gives the following.

$$
\begin{aligned}
a\dot{b}cde\dot{b}fe^{-1}gc^{-1}fa^{-1}gd &\sim a(bb)\dot{e}^{-1}d^{-1}c^{-1}f\dot{e}^{-1}gc^{-1}fa^{-1}gd \\
&\sim (bb)a(e^{-1}e^{-1})f^{-1}c\dot{d}gc^{-1}fa^{-1}g\dot{d} \\
&\sim (bb)(e^{-1}e^{-1})\dot{a}f^{-1}c(dd)g^{-1}\dot{a}f^{-1}cg^{-1} \\
&\sim (bb)(e^{-1}e^{-1})(dd)(aa)gc^{-1}ff^{-1}cg^{-1} \\
&\sim (bb)(e^{-1}e^{-1})(dd)(aa) \\
&\sim x_1x_1x_2x_2x_3x_3x_4x_4
\end{aligned}
$$

Hence the surface is non-orientable of genus 4.