Spring Semester 2015–2016

MATH31072/MATH41072/MATH61072 *Algebraic Topology*

Solutions 3

1. Take the number of triangle f. Note, that every triangle contains 3 edges. But in the the number $3f$ every edge got counted twice, since it occurs in exactly two triangles (by the link condition). Hence, we obtain $3f = 2e$. Plugging $\frac{3}{2}f$ for e into the formula $\chi = v - e + f$ gives the desired result.

2. Suppose that the triangle removed from K_1 is $\langle v_1, v_2, v_3\rangle$ and the triangle removed from K_2 is $\langle v_1', v_2', v_3'\rangle$ and then the vertices $v_i \sim v_i'$ are identified. Since there is an edge path in K_1 from v_1 to all other vertices and an edge path in K_2 from v_1' to all other vertices, it follows that K is connected.
For the link condition we need to check the link condition for the vertices $[v_i] = [v_i']$ for $1 \leqslant i \leqslant 3$. For $[v_1]$ suppose that the link of v_1 in K_1 is given by $\{\,\langle v_2, v_3\rangle,\ \langle v_3, v_4\rangle,\ \dots,\ \langle v_n, v_2\rangle\,\}$ (it must have this form by the link condition) and suppose that the link of v_1' in K_2 is given by $\{\,\langle v_2', v_3'\rangle,\ \langle v_3', v_4'\rangle,\ \dots,\ \langle v_m,' v_2'\rangle\,\}$. Now the link of $[v_1]$ in K includes all of the edges in both of these links apart from $\langle [v_2], [v_3]\rangle = \langle [v_2'], [v_3']\rangle$ which is not included since the triangles $\langle v_1, v_2, v_3\rangle$ and $\langle v_1', v_2', v_3'\rangle$ are removed. Hence the link of $[v_1]$ in K is given by $\{\,\langle [v_3], v_4\rangle,\ \dots,\ \langle v_n, [v_2]\rangle,\ \langle [v_2], v_m\rangle,\ \langle v_m, v_{m-1}\rangle, \dots,\ \langle v_4', [v_3']\rangle\,\}$ which is a simple closed polygon since $[v_3] = [v_3']$. The link condition for the other two identified vertices may be checked similarly. For all other vertices the link is unchanged. Hence K satisfies the link condition and so is a simplicial complex.

3. We prove that $\chi(T_g) = 2-2g$ by induction on g. The result is true for $g = 1$ by Example 3.3(b). For the inductive step suppose that the result is true for $g = k \geqslant 1$ so that $\chi(T_k) = 2-2k$. Then for $g = k+1$, $\chi(T_{k+1}) = \chi(T_k)+\chi(T_1)-2$ (by Proposition 3.4 and the inductive definition of T_g) $= (2-2k)+0-2 = 2-2(k+1)$ as required to prove the result for $g = k+1$. Hence, by induction on g the result holds for all $g \geqslant 1$.
Similarly for $\chi(P_g) = 2-g$. The result is true for $g = 1$ by Example 3.3(c). For the inductive step suppose that the result is true for $g = k \geqslant 1$ so that $\chi(P_k) = 2-k$. Then for $g = k+1$, $\chi(P_{k+1}) = \chi(P_k) + \chi(P_1) - 2$ (by Proposition 3.4 and the inductive definition of P_g) $= (2-k)+1-2 = 2-(k+1)$ as required to prove the result for $g = k+1$. Hence, by induction on g the result holds for all $g \geqslant 1$.

4. Suppose that K is a simplicial surface for which all but one of the triangle are oriented so that each pair of triangles with a common edge are coherently oriented. Suppose that the remaining triangle is $\langle v_1, v_2, v_3\rangle$. Suppose that the link of v_1 consists of the edges $\{\,\langle v_2, v_3\rangle,\ \langle v_3, v_4\rangle,\ \dots,\ \langle v_n, v_2\rangle\,\}$ (it must have this form by the link condition). Suppose that the triangle $\langle v_1, v_3, v_4\rangle$ is oriented as $\langle v_1, v_3, v_4\rangle$. This has the oriented edge $\langle v_1, v_3\rangle$. Then this triangle also has the oriented edge $\langle v_4, v_1\rangle$. Coherence across this edge means gives the orientation of the other triangle with this edge as $\langle v_1, v_4, v_5\rangle$. Continuing round the link we obtain the oriented triangles $\langle v_1, v_5, v_6\rangle$, etc. until we reach $\langle v_1, v_n, v_2\rangle$. This has the oriented edge $\langle v_2, v_1\rangle$. A similar argument using the link of v_2 (which

contains $\langle v_1 v_n \rangle$) tells us that the other triangle containing $\langle v_2, v_3 \rangle$ must orient that edge $\langle v_3, v_2 \rangle$. Hence, if we orient the triangle $\langle v_1, v_2, v_3 \rangle$ as $\langle v_1, v_2, v_3 \rangle$ then this is coherent with the orientations of the adjacent triangles and so gives us an orientation of K.
In this argument we started with the oriented triangle $\langle v_1, v_3, v_4 \rangle$. Had we started with the opposite orientation $\langle v_4, v_2, v_1 \rangle$ then all of the orientations in the above argument would have been reversed and the orientation $\langle v_3, v_2, v_1 \rangle$ of the triangle $\langle v_1, v_2, v_3 \rangle$ would have given us the required orientation of K.

5. (a) To calculate the Euler characteristic observe that the number of vertices $v = 9$ and the number of triangles $f = 18$. You can count, the number edges but it is easier to observe that since each triangle has three edges and each edge lies on two triangles $e = 3f/2 = (3 \times 18)/2 = 27$. Hence $\chi(K) = 9 - 27 + 18 = 0$. If each triangle in the diagram to the solution of Problems 8, Question 2(a) is oriented anticlockwise then coherence across each edge internal to the polygon is automatic and you can check coherence across the edges in the boundary of the polygon; hence K is orientable. Hence the surface is orientable of Euler characteristic 0 and so is homeomorphic to the torus.

(b) This surface has $v = 4$, $f = 4$ and so $e = (4 \times 3)/2 = 6$ and so $\chi(K) = 4 - 6 + 4 = 2$. Hence $|K|$ is homeomorphic to the sphere. (There is no need to check orientability since S^2 is the only closed surface with Euler characteristic 2. Orientability is automatic.)

(c) This surface has $v = 6$, $f = 10$ and so $e = (10 \times 3)/2 = 15$ and so $\chi(K) = 6 - 15 + 10 = 1$. Hence $|K|$ is homeomorphic to the projective plane. (There is no need to check orientability since P^2 is the only closed surface with Euler characteristic 1. Non-orientability is automatic.)

(d) This surface has $v = 7$, $f = 14$ and so $e = (14 \times 3)/2 = 21$ and so $\chi(K) = 7 - 21 + 14 = 0$. As with part (a) this is orientable and so $|K|$ is homeomorphic to the torus.