Autumn Semester 2016–2017

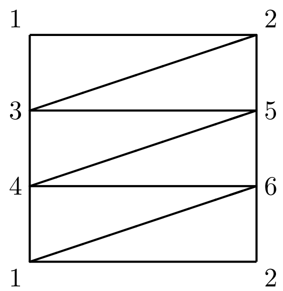
MATH41072/MATH61072 *Algebraic topology*

Solutions 4

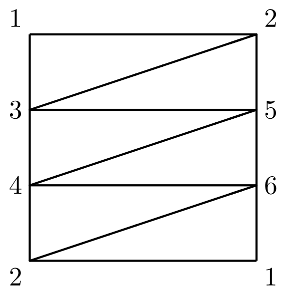
1. By the definition of the r-simplex $\langle v_0, v_1, \ldots, v_r\rangle$ every point of the simplex can be written in the required form. For uniqueness, suppose that $\sum_{i=0}^r t_i v_i = \sum_{i=0}^r t_i' v_i$ where $\sum_{i=0}^r t_i = \sum_{i=0}^r t_i' = 1$. Then $\sum_{i=0}^r (t_i - t_i')v_i = 0$ and $\sum_{i=0}^r (t_i - t_i') = 0$. Now write $s_i = t_i - t_i'$. Then $\sum_{i=0}^r s_i v_i = 0 \Rightarrow \sum_{i=1}^r s_i v_i = -s_0 v_0 = \sum_{i=1}^r s_i v_i \Rightarrow \sum_{i=1}^r s_i(v_i - v_0) = 0 \Rightarrow s_i = 0$ (for $1 \leqslant i \leqslant r$, since the set of vectors $v_i - v_0$ is linearly independent) $\Rightarrow s_0 = 0$. Hence $t_i = t_i'$ for $0 \leqslant i \leqslant r$ as required.

2. The function $|f|$ given in the statement of Corollary 4.11 is well defined by its definition and by the uniqueness of the barycentric coordinates. It is continuous on each simplex since it is a linear function on each simplex. The simplices are closed subsets of $|K|$ and so $|f|$ is continuous by the Gluing Lemma. It is a homeomorphism because the inverse $f^{-1}\colon K_2 \to K_1$ induces the inverse of $|f|$.

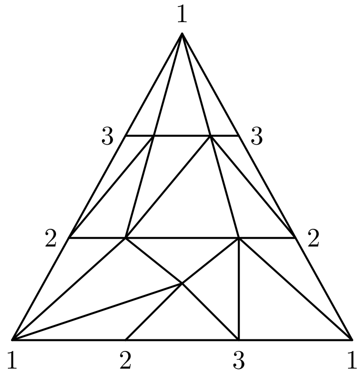
3. (a) A triangulation of the cylinder $I^2/(s,0) \sim (s,1)$ is obtained using the following template (using the arguments in Examples 2.8).



(b) Similarly, a triangulation of the Möbius band $I^2/(s,0) \sim (1-s,1)$ is obtained using the following template.



4. There are many possibilities here. It is important to ensure that in your template each 2-simplex is uniquely determined by its verties. One possible template is the following (where the five vertices inside the triangle) are all numbered differently.



5. (i) For $K = \bar{\Delta}^n$, $n_r = \binom{n+1}{r+1}$ and so $\chi(K) = \sum_{r=0}^{n}(-1)^r\binom{n+1}{r+1}$. To sum this series we can use the Binomial Theorem, $(1+x)^{n+1} = \sum_{i=0}^{n+1}\binom{n+1}{i}x^i$. For $x=-1$ this gives $0 = \sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i} = 1 - \sum_{r=0}^{n}(-1)^r\binom{n+1}{r+1}$ (putting $i = r+1$) and so $\chi(K) = 1$.

(ii) Using the notation of (i), for $L = K^{[n-1]}$, $\chi(L) = \sum_{r=0}^{n-1}(-1)^r n_r = \chi(K) - (-1)^n n_n = 1 + (-1)^{n+1}$ since $n_n = 1$.

(iii) Similarly, for $M = K^{[n-2]}$, $\chi(M) = \chi(K) - (-1)^n n_n - (-1)^{n-1}n_{n-1} = 1 + (-1)^{n+1} + (-1)^n(n+1)$ since $n_{n-1} = \binom{n+1}{n} = n+1$.

(iv) For my template, $n_0 = 8$, $n_1 = 24$ and $n_2 = 17$ and $\chi(K) = 8 - 24 + 17 = 1$. By the topological invariance, any other triangulation should also give $\chi(K) = 1$.

6. Following the hint a sequence of homeomorphisms is given as follows.

$\Delta^n \to \left\{(t_1,\ldots,t_n) \in \mathbb{R}^n \,|\, t_i \geqslant 0, \sum_{i=1}^{n} t_i \leqslant 1\right\}$ given by $(t_0,t_1,\ldots,t_n) \mapsto (t_1,\ldots,t_n)$. This is a bijection with continuous inverse since $t_0 = 1-(t_1+\ldots t_n)$.

$\left\{(t_1,\ldots,t_n) \in \mathbb{R}^n \,|\, t_i \geqslant 0, \sum_{i=1}^{n} t_i \leqslant 1\right\} \to I^n$ given by $\mathbf{t} \mapsto (\sum t_i)\mathbf{t}/(\max t_i)$ with inverse given by $\mathbf{t} \mapsto (\max t_i)\mathbf{t}/(\sum t_i)$.

$I^n \to [-1,1]^n$ given by $(t_1,\ldots,t_n) \mapsto (2t_1-1,\ldots,2t_n-1)$ with inverse given by
$(t_1,\ldots,t_n) \mapsto \big((t_1+1)/2,\ldots,(t_n+1)/2\big)$.

$[-1,1]^n \to D^n$ given by $\mathbf{t} \mapsto (\max|t_i|)\mathbf{t}/|\mathbf{t}|$ with inverse given by $\mathbf{t} \mapsto |\mathbf{t}|\mathbf{t}/(\max|t_i|)$.

[If anyone can see way of proving this result please let me know.]