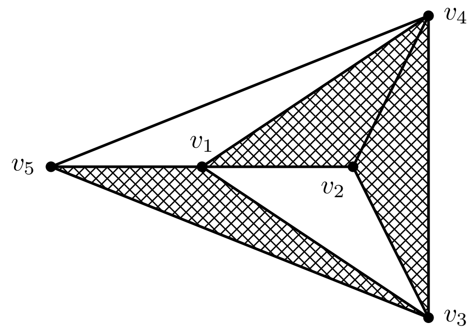
Autumn Semester 2017–2018

MATH41071/MATH61071 *Algebraic topology*

Solutions 5

1. The simplicial complex can be realized in the plane as follows.



By observation, $n_0 = 5$, $n_1 = 9$ and $n_2 = 3$ and so $\chi(K) = 5 - 9 + 3 = -1$.

$Z_0(K) = C_(K)$ is generated by $\langle v_1\rangle$, $\langle v_2\rangle$, $\langle v_3\rangle$, $\langle v_4\rangle$ and $\langle v_5\rangle$. $B_0(K)$ is generated by $\langle v_2\rangle - \langle v_1\rangle$, $\langle v_3\rangle - \langle v_1\rangle$, $\langle v_4\rangle$ and $\langle v_5\rangle - \langle v_1\rangle$. Hence $H_0(K) \cong \mathbb{Z}$ generated by $[\langle v_1\rangle] = [\langle v_2\rangle] = [\langle v_3\rangle] = [\langle v_4\rangle] = [\langle v_5\rangle]$. (An isomorphism $H_0(K) \to \mathbb{Z}$ is induced by the homomorphism $Z_0(K) \to \mathbb{Z}$ given by $\sum_i \lambda_i \langle v_i\rangle \mapsto \sum_i \lambda_i$.)

$Z_1(K) \cong \mathbb{Z}^5$ generated by $z_1 = \langle v_1, v_2\rangle + \langle v_2, v_3\rangle - \langle v_1, v_3\rangle$, $z_2 = \langle v_1, v_2\rangle + \langle v_2, v_4\rangle - \langle v_1, v_4\rangle$, $z_3 = \langle v_1, v_3\rangle + \langle v_3, v_5\rangle - \langle v_1, v_5\rangle$, $z_4 = \langle v_1, v_4\rangle + \langle v_4, v_5\rangle - \langle v_1, v_5\rangle$ and $z_5 = \langle v_2, v_3\rangle + \langle v_3, v_4\rangle - \langle v_2, v_4\rangle$. $B_1(K) \cong \mathbb{Z}^3$ generated by z_2, z_3 and z_5 and so $H_1(K) \cong \mathbb{Z}^2$ generated by $[z_1]$ and $[z_4]$.

$B_2(K) = Z_2(K) = 0$ and so $H_2(K) = 0$. $H_r(K) = 0$ for $r \neq 0, 1, 2$ since there are no r-simplices.

2. Let $f\colon H \to (H + K)/K$ be the homomorphism given by $f(h) = [h]$. This is an epimorphism since, given $[h + k] \in (H + K)/K$ where $h \in H$, $k \in K$, then $[h + k] = [h] = f(h)$. For the kernel of f, $h \in \operatorname{Ker}(f) \Leftrightarrow f(h) \in K \Leftrightarrow h \in K \Leftrightarrow h \in H \cap K$. Hence, by the First Isomorphism Theorem, f induces an isomorphism $\bar{f}\colon H/(H \cap K) \to (H + K)/K$ by $\bar{f}([h]) = f(h)$.

3. (i) In this case $H_0(K) \cong \mathbb{Z}$ since K is path-connected. $Z_1(K) = B_1(K) \cong \mathbb{Z}^3$ generated by $\langle v_1, v_2\rangle + \langle v_2, v_3\rangle - \langle v_1, v_3\rangle$, $\langle v_1, v_2\rangle + \langle v_2, v_4\rangle - \langle v_1, v_4\rangle$ and $\langle v_1, v_3\rangle + \langle v_3, v_4\rangle - \langle v_1, v_4\rangle$ and so $H_1(K) = 0$. $Z_2(K) = 0$ and so $H_2(K)$. All other homology groups are trivial for dimensional reasons (no non-empty r-simplices for $r \neq 0, 1, 2$).

(ii) $H_0(K) \cong \mathbb{Z}$ since K is connected. $H_1(K) = Z_1(K) \cong \mathbb{Z}^2$ generated by $\langle v_1, v_2\rangle + \langle v_2, v_3\rangle - \langle v_1, v_3\rangle$ and $\langle v_2, v_3\rangle + \langle v_3, v_4\rangle - \langle v_2, v_4\rangle$. All other homology groups are trivial for dimensional reasons.

(iii) $H_0(K) \cong \mathbb{Z}$ since K is connected. $H_1(K) \cong \mathbb{Z}^3$ generated by $\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle$, $\langle v_1, v_2\rangle + \langle v_2, v_3\rangle + \langle v_3, v_1\rangle$ $\langle v_0, v_1\rangle + \langle v_1, v_3\rangle + \langle v_3, v_0\rangle$. All other homology groups are trivial.

(iv) This is very similar to Example 4.33. Suppose we start with the simplicial complex in Examples 2.8(d). Then $Z_1(K) = B_1(K) + V$ where V is the free abelian group generated by

$$x = \langle v_1, v_2\rangle + \langle v_2, v_3\rangle + \langle v_3, v_4\rangle + \langle v_4, v_8\rangle - \langle v_5, v_8\rangle - \langle v_1, v_5\rangle.$$

Hence $H_1(K) = Z_1(K)/B_1(K) = (B_1(K) + V)/B_1(K) = V/\big(V \cap B_1(K)\big)$ by the Second Isomorphism Theorem. As before, if $d_2(z) \in V$ then z must be a multiple of $y = \langle v_1, v_2, v_5\rangle + \dots$ (all the 2-simplices oriented clockwise). But $d_2(y) = 2x$. Hence $V \cap B_1(K) \cong \mathbb{Z}$ generated by $2x$. Hence $H_1(K) \cong \mathbb{Z}_2$ generated by $[x]$.
For $z \in Z_2(K)$, z must be a multiple of y but since $d_2(y) \neq 0$ if follows that $Z_2(K) = 0$ and $H_2(K) = 0$.
$H_0(K) \cong \mathbb{Z}$ since K is connected and all other homology groups are trivial for dimensional reasons.

(v) Suppose that we use the simplicial complex K in the solution to Problems 4, Question 3(b). Then $H_0(K) = \mathbb{Z}$ since K is connected. By the usual argument, $Z_1(K) = B_1(K) + V$ where V is the free abelian group generated by

$$x_1 = \langle v_1, v_3\rangle + \langle v_3, v_4\rangle - \langle v_2, v_4\rangle - \langle v_1, v_2\rangle$$

and

$$x_2 = \langle v_1, v_2\rangle + \langle v_2, v_5\rangle + \langle v_5, v_6\rangle - \langle v_1, v_6\rangle.$$

Hence $H_1(K) = Z_1(K)/B_1(K) = (B_1(K) + V)/B_1(K) = V/\big(V \cap B_1(K)\big)$ by the Second Isomorphism Theorem. As before, if $d_2(z) \in V$ then z must be a multiple of $y = \langle v_1, v_2, v_3\rangle + \dots$ (all the 2-simplices oriented clockwise). But $d_2(y) = x_2 - x_1$. Hence $V \cap B_1(K) \cong \mathbb{Z}$ generated by $x_2 - x_1$. Hence $H_1(K) \cong \mathbb{Z}$ generated by $[x_1] = [x_2]$. $H_2(K) = Z_2(K) = 0$ since $d_2(y) \neq 0$ and all other homology groups are also trivial for dimensional reasons.

(vi) Suppose that we use the simplicial complex K in the solution to Problems 4, Question 3(a). Then $H_0(K) = \mathbb{Z}$ since K is connected. By the usual argument, $Z_1(K) = B_1(K) + V$ where V is the free abelian group generated by

$$x_1 = \langle v_1, v_3\rangle + \langle v_3, v_4\rangle - \langle v_1, v_4\rangle$$

and

$$x_2 = \langle v_2, v_5\rangle + \langle v_5, v_6\rangle - \langle v_2, v_6\rangle.$$

Hence $H_1(K) = Z_1(K)/B_1(K) = (B_1(K) + V)/B_1(K) = V/\big(V \cap B_1(K)\big)$ by the Second Isomorphism Theorem. As before, if $d_2(z) \in V$ then z must be a multiple of $y = \langle v_1, v_2, v_3\rangle + \dots$ (all the 2-simplices oriented clockwise). But $d_2(y) = x_2 - x_1$. Hence $V \cap B_1(K) \cong \mathbb{Z}$ generated by $x_2 - x_1$. Hence $H_1(K) \cong \mathbb{Z}$ generated by $[x_1] = [x_2]$. $H_2(K) = Z_2(K) = 0$ since $d_2(y) \neq 0$ and all other homology groups are also trivial for dimensional reasons.

4. Suppose that K is orientable. Then we may choose an orientations for all the triangles which are coherent across all the edges. Put $z =$ the sum of these oriented triangles. Then, coherence means that $d_2(z) = 0$ and so $z \in Z_2(\bar{K}) = H_2(\bar{K})$ and so $H_2(\bar{K}) \neq 0$.

Conversely, suppose that $H_2(\bar{K}) \neq 0$. Suppose that $z \in H_2(\bar{K}) = Z_2(\bar{K})$ is a non-trivial element. Then, since each edge appears in precisely two trianges, z is a non-trivial multiple of a cycle of the form $y = \sum \sigma_i$ where the σ_i are oriented triangles. Since $d_2(y) = 0$ the triangles in y are oriented coherently. However, y must involve all of the 2-simplices of K because K is connected and satisfies the link condition. For if there is some triangle of K not included in y then some vertex of a triangle in y must be s vertex of a triangle not included in y and then the link of this vertex would not be a simple closed polygon. Hence, all of the triangles of K may be oriented coherently and, furthermore, $H_2(\bar{K}) = Z_2(\bar{K}) \cong \mathbb{Z}$ generated by y.

5.

(i). It's enough to check this for simplices $\Delta = \langle v_0, \dots, v_r \rangle$. There are two cases to consider: Δ contains v as a vertex or not. Lets assume the latter, then

$$\begin{aligned} h(d(\Delta)) &= h(\sum (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_r \rangle) \\ &= \sum (-1)^i \langle v, v_0, \dots, \hat{v}_i, \dots, v_r \rangle \\ &= -d(\langle v, v_0, \dots, v_r \rangle) + \langle v_0, \dots, v_r \rangle \\ &= -d(h(\Delta)) + \Delta. \end{aligned}$$

Let's now assume Δ has v as a vertex, i.e. $v_j = v$ for some j. Then $d(h(\Delta)) = d(0) = 0$ and

$$\begin{aligned} h(d(\Delta)) &= h(\sum (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_r \rangle) \\ &= (-1)^j \langle v, v_0, \dots, \hat{v}_j, \dots, v_r \rangle \\ &= \langle v_0, \dots, v_r \rangle \\ &= \Delta. \end{aligned}$$

(ii). We have to show that every r-cycle is an r-boundary. Assume that $x \in Z_r(K)$. Then $d(x) = 0$ and $h(d(x)) = 0$. On the other hand, we have $d(h(x)) + h(d(x)) = x$. Hence, x is a boundary, since $x = d(h(x))$.